§6.7 Systems of ODEs

$A_{n\times n}$ — constant matrix

$\mathcal{L}\{y'\} = sY - y(0)$

$$\vec{y}'(t) = A\vec{y}(t) + \vec{g}(t)$$

$$\mathcal{L}\{\vec{y}'\} = s\vec{Y}(s) - \vec{y}(0)$$

$$\mathcal{L}\{A\vec{y}\} = A\vec{Y}(s)$$

$$s\vec{Y} - \vec{y}(0) = A\vec{Y} + \vec{G}$$

$$(sI - A)\vec{Y} = \vec{G} + \vec{y}(0)$$

$$\boxed{\vec{Y} = (sI-A)^{-1}(\vec{G} + \vec{y}(0))}$$

Ex.1 Mixing Problems

$$\vec{y}' = \begin{bmatrix} -0.08 & 0.02 \\ 0.08 & -0.08 \end{bmatrix} \vec{y}' + \begin{bmatrix} 6 \\ 0 \end{bmatrix}$$

$$\vec{G} = \mathcal{L}\left\{\begin{bmatrix} 6 \\ 0 \end{bmatrix}\right\} = \begin{bmatrix} \frac{6}{s} \\ 0 \end{bmatrix}$$

$$\vec{y}(0) = \begin{bmatrix} 0 \\ 150 \end{bmatrix}$$

$$sI - A = \begin{bmatrix} s+0.08 & -0.02 \\ -0.08 & s+0.08 \end{bmatrix}$$

$$(sI-A)^{-1} = \frac{1}{(s+0.08)^2 - 16\times 10^{-4}} \begin{bmatrix} s+0.08 & 0.02 \\ 0.08 & s+0.08 \end{bmatrix}$$

$$\vec{Y}(s) = \frac{1}{(s+a)^2 - b^2} \begin{bmatrix} s+0.08 & 0.02 \\ 0.08 & s+0.08 \end{bmatrix} \left(6 \begin{bmatrix} \frac{1}{s} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 150 \end{bmatrix} \right)$$

$a=0.08$, $b=0.04$

$$= \frac{1}{(s+0.12)(s+0.04)} \left\{ 6 \begin{bmatrix} \frac{s+0.08}{s} \\ \frac{0.08}{s} \end{bmatrix} + 150 \begin{bmatrix} 0.02 \\ s+0.08 \end{bmatrix} \right\}$$

$$= \frac{6}{s(s+0.12)(s+0.04)} \begin{bmatrix} s+0.08 \\ 0.08 \end{bmatrix} + \frac{150}{(s+0.12)(s+0.04)} \begin{bmatrix} 0.02 \\ s+0.08 \end{bmatrix}$$

. . . .

Ex.3 Two Masses on Springs

$$\begin{cases} y_1'' = -ky_1 + k(y_2 - y_1) \\ y_2'' = -k(y_2 - y_1) - ky_2 \end{cases}$$

$$\vec{y}'' = A\vec{y}, \quad A = \begin{bmatrix} -2k & k \\ k & -2k \end{bmatrix}$$

$$\vec{y}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \vec{y}'(0) = \cancel{\begin{bmatrix} \sqrt{3k} \\ -\sqrt{3k} \end{bmatrix}} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\mathcal{L}\{\vec{y}''\} = s^2\vec{Y}(s) - s\vec{y}(0) - \vec{y}'(0) = A\vec{Y}$$

$$(s^2 I - A)\vec{Y} = s\vec{y}(0) + \cancel{\vec{y}'(0)}$$

$$\begin{bmatrix} s^2+2k & -k \\ -k & s^2+2k \end{bmatrix}^{-1} = \frac{1}{(s^2+2k)^2 - k^2} \begin{bmatrix} s^2+2k & k \\ k & s^2+2k \end{bmatrix}$$

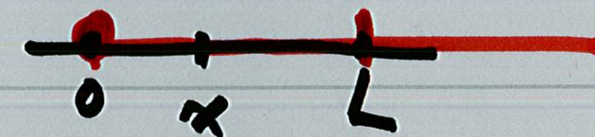
$$a^2 - b^2 = (a+b)(a-b)$$

$$\vec{Y} = \frac{1}{(s^2+3k)(s^2+k)} \begin{bmatrix} s^2+2k & k \\ k & s^2+2k \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} s$$

$$= \frac{\cancel{1}\, s}{(s^2+3k)(s^2+k)} \begin{bmatrix} s^2+3k \\ s^2+3k \end{bmatrix} = \frac{s}{s^2+k} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\vec{y}(t) = \mathcal{L}^{-1}\{\vec{Y}\} = \mathcal{L}\left\{\frac{s}{s^2+k}\right\} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \cos\sqrt{k}\,t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

§12.12 Solution of PDEs by Laplace Transforms



$w(L,t)=g(t)$

- wave equation

$$\frac{\partial^2 w(x,t)}{\partial t^2} = c^2 \frac{\partial^2 w}{\partial x^2}, \qquad (0,+\infty)\times\boxed{(0,+\infty)}$$

$x \in [0, L]$

"boundary conditions"

$$\begin{cases} w(0,t) = f(t) \\ \lim\limits_{x\to\infty} w(x,t) = 0 \end{cases}$$

initial conditions

$$\begin{cases} w(x,0) = 0 \\ w_t(x,0) = 0 \end{cases}$$

(1) Laplace transform w.r.t. t $\qquad W(x,s) = \mathcal{L}\{w(x,t)\}$

ODE $\quad \boxed{\dfrac{\partial^2 W(x,s)}{\partial x^2} - \dfrac{s^2}{c^2} W = 0}$

ODE $\quad \mathcal{L}\{y''\} = s^2 Y - s\,y(0) - y'(0)$

$$\mathcal{L}\left\{\frac{\partial^2 w}{\partial t^2}\right\} = s^2 W(x,s) - s\,w(x,0) - w_t(x,0) = s^2 W(x,s)$$

$$\mathcal{L}\left\{\frac{\partial^2 w}{\partial x^2}\right\} = \int_0^\infty e^{-st}\frac{\partial^2 w}{\partial x^2}\,dt = \frac{\partial^2}{\partial x^2}\int_0^\infty e^{-st} w(x,t)\,dt \boxed{= \frac{\partial^2}{\partial x^2} W(x,s)}$$

(2) General Solution of ODE

$$W'' - \boxed{\frac{s^2}{c^2}} W = 0 \qquad r^2 - \frac{s^2}{c^2} = 0$$

constant

$$r = \pm \sqrt{\frac{s}{c}}$$

$$W = e^{rt}$$

$$W(x,s) = A(s)\,e^{\frac{sx}{c}} + B(s)\,e^{-\frac{sx}{c}}$$

$e^{\frac{sx}{c}} \to \infty$, $e^{-\frac{sx}{c}} \to 0$

(3) Using BCs to Determine $\boxed{A(s) = 0 \text{ and } B(s) = F(s)}$ $\Longrightarrow$ $W(x,s) = F(s)\,e^{-\frac{sx}{c}}$

$$w(0,t) = f(t) \qquad W(0,s) = \mathcal{L}\{w(0,t)\} = F(s) = A(s) + B(s)$$

$$\lim_{x\to\infty} w(x,t) = 0 \quad \lim_{x\to\infty} W(x,s) = 0 = A(s)\cdot\infty \Longrightarrow A(s) = 0$$

(4) Solution of PDE

$$w(x,t) = \mathcal{L}^{-1}\{W(x,s)\}$$