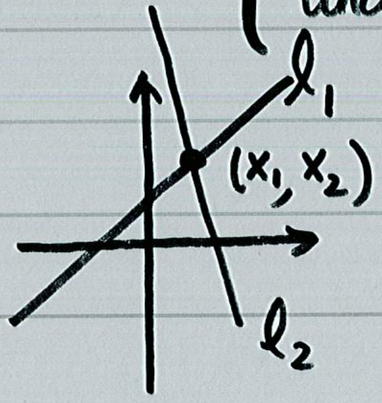
§7.3 Linear System of Eqs. Gauss Elimination.

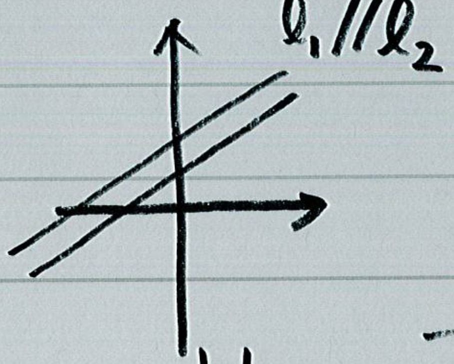
$$A\vec{x}=\vec{b}. \qquad A=\underbrace{\begin{bmatrix}a_{11} & \cdots & a_{1n}\\ \vdots & & \\ a_{m1} & \cdots & a_{mn}\end{bmatrix}}_{\text{coefficient matrix}},\quad \underbrace{\vec{x}=\begin{bmatrix}x_1\\ \vdots\\ x_n\end{bmatrix}}_{\text{unknowns}},\quad \vec{b}=\begin{bmatrix}b_1\\ \vdots\\ b_m\end{bmatrix}$$

- $\begin{cases}\text{homogeneous} & \vec{b}=\vec{0}\\ \text{non-homog.} & \vec{b}\neq\vec{0}\end{cases}$
- $\begin{cases}\text{overdetermined} & m>n\\ \text{determined} & m=n\\ \text{underdetermind} & m<n\end{cases}$
- $\begin{cases}\text{consistent} & \text{at least one solution}\\ \text{inconsistent} & \text{no solutions}\end{cases}$

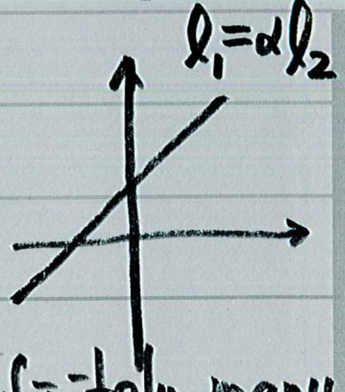
$$\begin{aligned} l_1:&\begin{cases}a_{11}x_1+a_{12}x_2=b_1\\ a_{21}x_1+a_{22}x_2=b_2\end{cases}\\ l_2:& \end{aligned}$$



exact one solution



no solutions



infinitely many solutions

Gauss Elimination and Back Substitution

$$A_{m\times n}\vec{x}=\vec{b}.$$

$$A_{m\times n}=\begin{bmatrix}\vec{a}_1\\ \vdots\\ \vec{a}_m\end{bmatrix}$$ $\vec{a}_i$ — the i^{th}-row vector of A

- Elementary Row Operation (ERO)

(1) $\vec{a}_j \longleftrightarrow \vec{a}_k$; (2) $\alpha\vec{a}_j+\vec{a}_k$; (3) $\alpha\vec{a}_j$ with $\alpha\neq 0$

- systems S_1 & S_2 are row-equivalent $\iff S_1 \xrightarrow{ERO} S_2$

- Row-equivalent systems have the same set of solutions.

$[A\,|\,\vec{b}\,]$ augmented matrix

- $\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ -1 & 1 & 1 & 0 \\ 0 & 10 & 25 & 90 \\ 20 & 10 & 0 & 80 \end{array}\right]$

- $\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 2 & 2 & 4 \end{array}\right]$

- $\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 2 & 2 & 2 \end{array}\right]$

$$[A|\vec{b}] \xrightarrow{ERO} [R|\vec{f}] = \left[\begin{array}{cccccc|c} r_{11} & r_{12} & \cdots & \cdots & \cdots & r_{1n} & f_1 \\ & r_{22} & \cdots & \cdots & \cdots & r_{2n} & \vdots \\ & 0 & \ddots & & & \vdots & \\ & & & r_{rr} & \cdots & r_{rn} & f_r \\ \hline & & & & & & f_{r+1} \\ & & & 0 & & & \vdots \\ & & & & & & f_m \end{array}\right]$$