§7.4 Linear Independence. Rank. Vector Space.

$\vec{a}_1, \cdots, \vec{a}_m$ m vectors with the same # of components

- **linear combination** $c_1\vec{a}_1 + \cdots + c_m\vec{a}_m$, $c_i \in R$

- **linearly indep.** $c_1\vec{a}_1 + \cdots + c_m\vec{a}_m = \vec{0} \implies c_1 = \cdots = c_m = 0$

$\{\vec{a}_1, \vec{a}_2\} = \{(1,0),(0,1)\}$ l. indep.

- **linear span** $\operatorname{span}\{\vec{a}_1, \cdots, \vec{a}_m\} = \left\{ \sum_{i=1}^{m} c_i\vec{a}_i \,\middle|\, c_i \in R \right\}$

Rank of Matrix

$$A_{m\times n} = \begin{bmatrix} \vec{a}_1 \\ \vdots \\ \vec{a}_m \end{bmatrix}$$ $\vec{a}_i$ — i^{th} row vector of A

- $r(A)$ = the max. # of l. indep. row vectors of A

Properties (1) $r(A)=m \Longleftrightarrow \{\vec{a}_1, \cdots, \vec{a}_m\}$ is l. indep.

(2) $r(A) < m \Longleftrightarrow \{\vec{a}_1, \cdots, \vec{a}_m\}$ is l. dep

(3) $r(A) = r(A^t)$ = the max # of l. indep column vectors

(4) $\{\vec{a}_1, \cdots, \vec{a}_m\}$ & $\vec{a}_i \in R^n$: $n < m \Rightarrow \{\vec{a}_1, \cdots, \vec{a}_m\}$ l. dep.

$$RS_A = \text{span}\left\{\text{rows of } \underset{m\times n}{A} = \begin{bmatrix} \vec{a}_1 \\ \vdots \\ \vec{a}_m \end{bmatrix}\right\} = \left\{ \sum_{i=1}^{m} c_i \vec{a}_i \;\middle|\; c_i \in R \right\}$$

$$CS_A = \text{span}\left\{\text{columns of } \underset{m\times n}{A} = [\vec{c}_1, \cdots, \vec{c}_n]\right\} = \left\{ \sum_{i=1}^{n} \alpha_i \vec{c}_i \;\middle|\; \alpha_i \in R \right\}$$

$$N_A = \left\{ \vec{x} \in R^n \;\middle|\; \underset{m\times n}{A} \underset{n\times 1}{\vec{x}} = \underset{m\times 1}{\vec{0}} \right\}$$

<u>Properties</u> (1) $\dim RS_A = \dim CS_A = r(A)$

(2) $\dim N_{\underset{m\times n}{A}} = n - r(A)$

#3, 19, 27, 29 on p.287

#3 on P. 287

$$\begin{bmatrix} 0 & 3 & 5 \\ 3 & 5 & 0 \\ 5 & 0 & 10 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & 3 & 5 \\ 3 & 5 & 0 \\ 0 & -\frac{25}{3} & 10 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & 3 & 5 \\ 3 & 5 & 0 \\ 0 & 0 & \frac{215}{9} \end{bmatrix}$$

$$RS_A = \text{span}\{[0,3,5], [3,5\ 0], [5\ 0\ 10]\} \overset{?}{=} \text{span}\{[1,0,0], [0,1,0], [0,0,1]\}$$

$$CS_A = \text{span}\{[0,3,5]^t, [3,5,0]^t, [5,0,10]^t\}$$

#19 on P. 287

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ l. indep.

#29 on P. 288 $\quad V=\left\{\vec{v}\in R^2 \mid v_1 \geqslant v_2\right\}$

V is not a vector space: e.g., $\begin{bmatrix}2\\1\end{bmatrix}\in V$ since $2 \geqslant 1$

but $-\begin{bmatrix}2\\1\end{bmatrix}\notin V$ since $-2<-1$

#27 on P. 288 $\quad V=\left\{\vec{v}\in R^3 \mid v_1-v_2+2v_3=0\right\}=\left\{\vec{v}\in R^3 \mid [1,-1,2]\,\vec{v}=0\right\}$

$A=[1,-1,2],\ \forall \vec{v},\vec{w}\in V \Rightarrow A\vec{v}=A\vec{w}=0$

$\Rightarrow A(\alpha\vec{v}+\beta\vec{w})=\alpha A\vec{v}+\beta A\vec{w}=0 \Rightarrow V$ is a vector space

Let $v_2=s$, $v_3=t$ $\Rightarrow \begin{cases} v_1=s-2t \\ v_2=s \\ v_3=t \end{cases} \Rightarrow \vec{v}=s\begin{bmatrix}1\\1\\0\end{bmatrix}+t\begin{bmatrix}-2\\0\\1\end{bmatrix} \Rightarrow V=\operatorname{span}\left\{\begin{bmatrix}1\\1\\0\end{bmatrix},\begin{bmatrix}-2\\0\\1\end{bmatrix}\right\}$

$\dim V=2$