§7.5 Solutions of Linear Systems

$$A_{m\times n}\,\vec{x}_{n\times 1}=\vec{b}_{n\times 1}$$

m — # of equations

n — # of unknowns

Guass Elimination

$$[A\,|\,\vec{b}\,]\xrightarrow{\text{ERO}}[R\,|\,\vec{f}\,]=\left[\begin{array}{ccccc|c} r_{11} & r_{12} & \cdots & \cdots & r_{1n} & f_1 \\ & r_{22} & \cdots & \cdots & r_{2n} & f_2 \\ & 0 & \ddots & & \vdots & \vdots \\ & & & r_{rr} \cdots & r_{rn} & f_r \\ & & & & & f_{r+1} \\ & & 0 & & & \vdots \\ & & & & & f_m \end{array}\right]$$

$r(A)=r$

# of nonzero rows

- **consistent** "$r=m$" or "$r<m$ and $[f_{r+1},\cdots,f_m]^t=\vec{0}$" $r(A)=r(A|\vec{b})$
- **inconsistent** $r<m$ and $[f_{r+1},\cdots,f_m]^t\neq\vec{0}$ $r(A)<r(A|\vec{b})$
- **unique solution** consistent and $r(A)=n$ $r(A)=r(A|\vec{b})=n$
- **infinitely many solutions** consistent and $r(A)<n$ $r(A)=r(A|\vec{b})<n$

$$A_{m\times n}\vec{x}_{n\times 1} = \vec{b}_{m\times 1}$$

- <u>view based on column vectors of A</u>

$$A = [\vec{c}_1\ \vec{c}_2\ \cdots\ \vec{c}_n] \Longrightarrow \vec{b} = x_1\vec{c}_1 + x_2\vec{c}_2 + \cdots + x_n\vec{c}_n$$

- <u>inconsistent</u> $\vec{b} \notin \operatorname{span}\{\vec{c}_1, \vec{c}_2, \cdots, \vec{c}_n\} = CS_A$ $\quad r(A) < r(A|\vec{b})$

- <u>consistent</u> $\vec{b} \in CS_A$ $\quad r(A) = r(A|\vec{b})$

- <u>unique solution</u> $\vec{b} \in CS_A$ and $r(A) = n$ $\quad \begin{cases} A\vec{x} = \vec{b} \\ A\vec{y} = \vec{b} \end{cases} \Rightarrow A(\vec{x} - \vec{y}) = \vec{0}$

$$\Rightarrow \vec{x} - \vec{y} = \vec{0}$$

- <u>infinitely many solutions</u> $\vec{b} \in CS_A$ and $r(A) < n$

$$A_{m\times n}\,\vec{x}_{n\times 1} = \vec{b}_{m\times 1}$$ homogeneous system

- <u>trivial solution</u> $\quad \vec{x}_{n\times 1} = \vec{0}_{n\times 1}$
- <u>non-trivial solutions</u> $\quad \vec{x} \neq \vec{0}$ and $A\vec{x} = \vec{0}$ $\qquad r(A) < n$
- <u>solution space</u> $\quad N_A = \left\{ \vec{x} \in R^n \;\middle|\; A\vec{x} = \vec{0} \right\}$

<u>nullity of A</u> $\quad \dim N_A = n - r(A)$

$$A = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & -4 \\ 0 & 4 & 0 \end{bmatrix}$$

$N_A = ?$ $\quad \dim N_A = ?$

$$A_{m\times n}\vec{x}_{n\times 1} = \vec{b}_{m\times 1}$$ non-homogeneous system

- <u>general solution</u> assume $r(A) = r(A|\vec{b})$

$$\boxed{\vec{x} = \vec{x}_0 + \vec{x}_h}$$

$\vec{x}_0$ — a particular solution

$\vec{x}_h$ — general solutions of $A\vec{x} = \vec{0}$

$$\forall \vec{x} \text{ s.t. } A\vec{x} = \vec{b} \Rightarrow A(\vec{x} - \vec{x}_0) = \vec{0} \Rightarrow \vec{x} - \vec{x}_0 = \vec{x}_h \Longrightarrow \vec{x} = \vec{x}_0 + \vec{x}_h$$

$$\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & -4 \\ 0 & 4 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}$$