§7.6-7.7 Determinants. Cramer Rules.

$$A_{n\times n} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}, \quad D = \det A = |A| = \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix}$$

$$|a_{11}| = a_{11}, \quad \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}, \quad \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{21}\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{31}\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$$

$D = a_{j1}C_{j1} + \cdots + a_{jn}C_{jn}$, $j = 1, 2, \cdots,$ **or** n $\quad$ j^{th}-row expansion

$C_{jk} = (-1)^{j+k} M_{jk}$

cofactor $\quad$ minor

M_{jk} — det of $(n-1)\times(n-1)$ submatrix by omitting j^{th}-row and k^{th}-column of A

Computation of $|A|$ by expansion

$$\begin{vmatrix} 0 & 4 & -1 \\ -4 & 0 & 3 \\ 1 & -3 & 0 \end{vmatrix} =$$

Computation of $|A|$ by ERO and expansion

$$A \xrightarrow{\text{ERO}} B$$

$$\underset{n\times n}{A} = \begin{bmatrix} \vec{a}_1 \\ \vec{a}_2 \\ \vdots \\ \vec{a}_n \end{bmatrix}$$

- $\vec{a}_j \longleftrightarrow \vec{a}_k$ $\quad -|A| = |B|$
- $\vec{a}_k + c\,\vec{a}_j$ $\quad |A| = |B|$
- $c\,\vec{a}_k \ (c \neq 0)$ $\quad c|A| = |B|$

$$\begin{vmatrix} 0 & 4 & -1 \\ -4 & 0 & 3 \\ 1 & -3 & 0 \end{vmatrix} =$$

Properties of Determinant

- $|A^t| = |A|$

- $A_{n\times n}$ has a zero row/column $\Longrightarrow |A| = 0$ $\quad \begin{vmatrix} 1 & 1 \\ 0 & 0 \end{vmatrix} = 0$

- $A_{n\times n}$ has proportional rows/columns $\Longrightarrow |A| = 0$ $\quad \begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix} = 0$

- $r(A_{m\times n}) = r \geq 1 \iff$ (1) $\exists$ submatrix $S_{r\times r}$ s.t. $|S_{r\times r}| \neq 0$
  (2) $\forall$ submatrix $S_{q\times q}$ $(q > r)$
  s.t. $|S_{q\times q}| = 0$

- $r(A_{n\times n}) = n \iff |A| \neq 0$

Cramer's Rule

$$A_{n\times n}\vec{x}_{n\times 1}=\vec{b}_{n\times 1} \quad \text{with } |A|\neq 0$$

unique solution $\quad \vec{x}=[x_1, x_2, \cdots, x_n]^t$

$$x_i=\frac{D_i}{D}, \quad D=|A| \text{ and } D_i=|A_i|$$

$$A=[\vec{c}_1, \cdots, \vec{c}_n], \quad A_i=[\vec{c}_1, \cdots, \vec{c}_{i-1}, \vec{b}, \vec{c}_{i+1}, \cdots, \vec{c}_n]$$

$\vec{b}=\vec{0} \qquad \vec{x}=\vec{0}$

$$\begin{bmatrix}3 & -5\\ 6 & 16\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}15.5\\ 5\end{bmatrix}$$