§7.9 Vector Spaces. Inner Product Spaces. Linear Transformation

Real Vector Space V $\quad$ V — non-empty set s.t.

- vector addition : $\forall \vec{a}, \vec{b} \in V \Longrightarrow \vec{a}+\vec{b} \in V$

  (1) $\vec{a}+\vec{b} = \vec{b}+\vec{a}$, $\quad$ (2) $(\vec{a}+\vec{b})+\vec{c} = \vec{a}+(\vec{b}+\vec{c})$,

  (3) $\exists\, \vec{0} \in V$ s.t. $\vec{a}+\vec{0}=\vec{a}$, (4) $\forall \vec{a} \in V$, $\exists\, -\vec{a} \in V$, s.t., $\vec{a}+(-\vec{a})=\vec{0}$.

- scalar multiplication: $\forall c \in R, \vec{a} \in V \Longrightarrow c\vec{a} \in V$

  (1) $c(\vec{a}+\vec{b}) = c\vec{a}+c\vec{b}$, (2) $(c+k)\vec{a} = c\vec{a}+k\vec{a}$,

  (3) $c(k\vec{a}) = (ck)\vec{a}$, $\quad$ (4) $1\cdot\vec{a} = \vec{a}$.

linear combination

$$c_1\vec{a_1} + c_2\vec{a_2} + \cdots + c_m\vec{a_m}\ , \quad c_i \in R$$

- $\{\vec{a_1}, \cdots, \vec{a_m}\}$ is linearly independent

$$c_1\vec{a_1} + \cdots + c_m\vec{a_m} = \vec{0} \implies c_1 = c_2 = \cdots = c_m = 0$$

- $\dim V$ = the max # of l. indep. vectors
- <u>basis of V</u> any $\dim V$ l. indep. vectors

$$V = \left\{ \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \middle| \ a_{ij} \in R \right\}$$

$\dim V = ?$ basis $= ?$ #7, 10 p318

#7 on P. 318

$$V = \{ y(x) = (ax+b)e^{-x} \mid a, b \in R \}$$

Solution V is a vector space: let $y_i(x) = (a_i x + b_i)e^{-x}$

$$c_1 y_1(x) + c_2 y_2(x) = [(c_1 a_1 + c_2 a_2)x + (b_1 + b_2)]e^{-x} \in V$$

because $c_1 a_1 + c_2 a_2$, $b_1 + b_2 \in R$

$$V = \{ a x e^{-x} + b e^{-x} \mid a, b \in R \} = \text{span}\{ xe^{-x}, e^{-x} \}$$

$\dim V = 2$, basis: xe^{-x}, e^{-x}

#10 on P. 318

$$V = \left\{ A_{3\times 2} = [\vec{a}_1, \vec{a}_2] \;\middle|\; \vec{a}_1 = c\begin{bmatrix}3\\0\\5\end{bmatrix}, \vec{a}_2 = \begin{bmatrix}a_{12}\\a_{22}\\a_{32}\end{bmatrix}, c, a_{j2} \in R \right\}$$

$$\forall\, A_1, A_2 \in V \implies \alpha A_1 + \beta A_2 = \left[\gamma\begin{bmatrix}3\\0\\5\end{bmatrix}, \vec{a}\right] \in V$$

$$\implies V \text{ is a vector space}$$

$$V = \text{span}\left\{ \begin{bmatrix}3&0\\0&0\\5&0\end{bmatrix}, \begin{bmatrix}3&1\\0&0\\5&0\end{bmatrix}, \begin{bmatrix}3&0\\0&1\\5&0\end{bmatrix}, \begin{bmatrix}3&0\\0&0\\5&1\end{bmatrix} \right\}$$

$$= \left\{ \begin{bmatrix}3c&a_{12}\\0&a_{22}\\5c&a_{23}\end{bmatrix} \;\middle|\; c, a_{j2} \in R \right\}$$

$$\dim V = 4$$

Inner Product Space V   V — real vector space

- inner product : $(\vec{a}, \vec{b})$

(1) $\forall c_i \in R, \forall \vec{a}, \vec{b} \in V \Longrightarrow (c_1\vec{a} + c_2\vec{b}, \vec{c}) = c_1(\vec{a}, \vec{c}) + c_2(\vec{b}, \vec{c})$

(2) $(\vec{a}, \vec{b}) = (\vec{b}, \vec{a})$,   (3) $(\vec{a}, \vec{a}) \geq 0$

$(\vec{a}, \vec{a}) = 0 \Longleftrightarrow \vec{a} = \vec{0}$

orthogonality $\vec{a} \perp \vec{b} \Longleftrightarrow (\vec{a}, \vec{b}) = 0$

norm $\|\vec{a}\| = \sqrt{(\vec{a}, \vec{a})}$

Inequalities $|(\vec{a}, \vec{b})| \leq \|\vec{a}\| \|\vec{b}\|$, $\|\vec{a} + \vec{b}\| \leq \|\vec{a}\| + \|\vec{b}\|$

$$R^n = \left\{ (x_1, \cdots, x_n) \;\middle|\; x_i \in R \right\} = \operatorname{span}\left\{ [1, 0, \cdots, 0], [0, 1, 0, \cdots, 0], \cdots, [0, \cdots, 0, 1] \right\}$$

$$\vec{a} \cdot \vec{b} = a_1 b_1 + \cdots + a_n b_n = \|\vec{a}\| \, \|\vec{b}\| \cos\langle \vec{a}, \vec{b} \rangle$$

$$\|\vec{a}\| = \sqrt{a_1^2 + \cdots + a_n^2}$$

$$C[\alpha, \beta] = \left\{ f(x) \;\middle|\; f \text{ is continuous on } [\alpha, \beta] \right\}$$

$$(f, g) = \int_\alpha^\beta f(x) g(x) \, dx$$

$$\|f\| = \sqrt{(f, f)} = \int_\alpha^\beta f(x)^2 \, dx$$

#21 (p318)

Linear Transformation

X, Y — vector spaces

$$F : X \to Y \text{ by } \vec{y} = F(\vec{x}) \text{ for } \vec{x} \in X$$

F is linear $\iff \begin{cases} F(\vec{x}+\vec{v}) = F(\vec{x}) + F(\vec{v}) \\ F(c\vec{x}) = cF(\vec{x}) \end{cases}$

- $F : R^n \to R^m$ is linear $\iff F(\vec{x}) = A_{m\times n}\vec{x}_{n\times 1} \in R^m$

Inverse Transformation (linear)

$F : R^n \to R^n$

$$\vec{y} = F(\vec{x}) = A_{n\times n}\vec{x} \xRightarrow{A^{-1}\text{ exists}} \vec{x} = F^{-1}(\vec{y}) = A^{-1}\vec{y}$$

#11 p318

#21 on p. 318

Find k s.t. $0 = \begin{bmatrix} 2 \\ \frac{1}{2} \\ -4 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ k \\ 0 \\ \frac{1}{4} \end{bmatrix} = 10 + \frac{k}{2} \Rightarrow k = -20$

#11 on p. 318 $\vec{y} = \begin{bmatrix} 0.5 & -0.5 \\ 1.5 & -2.5 \end{bmatrix} \vec{x} = A\vec{x}$

$$\Rightarrow \vec{x} = A^{-1}\vec{y}$$

$$A^{-1} = \frac{1}{|A|} \begin{bmatrix} -2.5 & +0.5 \\ -1.5 & 0.5 \end{bmatrix} = - \begin{bmatrix} -5 & 1 \\ -3 & 1 \end{bmatrix}$$