Chapter 8 Matrix Eigenvalue Problems (8.1, 8.3&8.5, 8.4)

§8.1 Matrix Eigenvalue Problems

λ — eigenvalue/characteristic value

$$A\vec{x} = \lambda\vec{x}$$

$0 \neq \vec{x}$ — eigenvector

spectrum of $A = \{\lambda \mid A\vec{x} = \lambda\vec{x}\}$

spectral radius of $A = \max\{|\lambda| \mid A\vec{x} = \lambda\vec{x}\}$

<u>Calculation</u> $\quad (\underset{n\times n}{A} - \lambda \underset{n\times n}{I})\vec{x} = \vec{0}$

(1) eigenvalues $\det(A - \lambda I) = 0$; (2) eigenvector $(A - \lambda_i I)\vec{x} = \vec{0}$

Examples

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 5 & -2 \\ 9 & -6 \end{bmatrix}, \quad A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

Properties

(1) $A_{n\times n}$ has at least one eigenvalue and at most n numerically different eigenvalues.

(2) $$\underbrace{\{\vec{x} \mid A\vec{x} = \lambda\vec{x}\} \cup \{\vec{0}\}}_{\text{eigenspace of } A} \text{ is a vector space.}$$

- $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $0 = |A - \lambda I| = \begin{vmatrix} -\lambda & 0 \\ 0 & -\lambda \end{vmatrix} = \underline{\underline{\lambda^2}} \Rightarrow \underline{\underline{\lambda_1 = \lambda_2 = 0}}$

$\underline{\lambda = 0}$ $\quad \underline{A\vec{x} = \vec{0}}$ $\quad \begin{matrix} 0 = 0 \\ 0 = 0 \end{matrix}$ $\quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} s \\ t \end{bmatrix} = \underline{s}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \underline{t}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\vec{x}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\vec{x}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

- $A = \begin{bmatrix} 5 & -2 \\ 9 & -6 \end{bmatrix}$ $\quad 0 = \det(A - \lambda I) = \det\begin{pmatrix} 5-\lambda & -2 \\ 9 & -6-\lambda \end{pmatrix}$

$$= (\lambda - 5)(\lambda + 6) + 18$$

$$= \lambda^2 + \lambda - 12 = (\lambda - 3)(\lambda + 4)$$

$\lambda_1 = -4$, $\lambda_2 = 3$

$\underline{\lambda_1 = \cancel{4}-4}$ $\quad (A+4I)\vec{x}=\vec{0}$ $\quad \begin{bmatrix} 9 & -2 \\ 9 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$9x_1 - 2x_2 = 0 \implies \begin{cases} x_1 = \frac{2}{9}x_2 = \frac{2}{9}s \\ x_2 = s \end{cases} \implies \vec{x}^{(1)} = s\begin{bmatrix} \frac{2}{9} \\ 1 \end{bmatrix}$

$\begin{bmatrix} 2 \\ 9 \end{bmatrix}$

$\underline{\lambda_2 = 3}$ $\quad (A-3I)\vec{x}=0$ $\quad \begin{bmatrix} 2 & -2 \\ 9 & -9 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$2x_1 - 2x_2 = 0 \implies \begin{cases} x_1 = x_2 = s \\ x_2 = s \end{cases} \implies \vec{x}^{(2)} = s\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

• $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$ $\quad 0 = \det(A - \lambda I) = \begin{vmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{vmatrix}$

$-(\lambda+2)$

2

$$= -\begin{vmatrix} 0 & 2\underline{(\lambda+3)} & \lambda^2+2\lambda-3 \\ 0 & -\underline{(\lambda+3)} & -2(\lambda+3) \\ -1 & -2 & -\lambda \end{vmatrix}$$

$2\lambda+4+2$

$\lambda(\lambda+2) - 3$

$$= +(\lambda+3)\begin{vmatrix} 2 & \lambda^2+2\lambda-3 \\ +1 & +2(\lambda+3) \end{vmatrix} = (\lambda+3)\left[4\lambda + \overset{12}{\cancel{8}} \overset{-}{\cancel{+}} (\lambda^2+2\lambda-3)\right]$$

$$= (\lambda+3)\left[-\lambda^2+2\lambda+15\right] = -(\lambda+3)(\lambda+3)(\lambda-5)$$

$\lambda^2-2\lambda-15$

$\lambda_1=\lambda_2=-3$, $\lambda_3=5$

<u>eigenvectors</u>

<u>$\lambda_1=\lambda_2=-3$</u>

$$\begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\vec{0}$$

$x_1+2x_2-3x_3=0$

set $x_2=s$, $x_3=t$ $\Longrightarrow \vec{x}=\begin{bmatrix} -2s+3t \\ s \\ t \end{bmatrix} = s\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}+t\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$

$$\vec{x}^{(1)}=\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \quad \vec{x}^{(2)}=\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$$

<u>$\lambda_3=5$</u>

$$\begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix}\vec{x}=\vec{0} \qquad \begin{bmatrix} -1 & -2 & -5 \\ 0 & -8 & \cancel{4}\,-16 \\ 0 & +16 & 32 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 5 \\ 0 & 8 & 16 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{cases} x_1+2x_2+5x_3=0 \\ x_2+2x_3=0 \end{cases} \xRightarrow{x_3=s} \vec{x}=\begin{bmatrix} -2(-2s)-5s \\ -2s \\ s \end{bmatrix}=s\begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}, \quad \vec{x}^{(3)}=\begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$$