§8.3 & 8.5 Symmetric (Hermitian), Orthogonal (Unitary) Matrices

$R = \{ c \mid c \text{ is a real number} \}$, $C = \{ a = \alpha + i\beta \mid \alpha, \beta \in R \}$, $\bar{a} = \alpha - i\beta$

$|a| = \sqrt{\alpha^2 + \beta^2}$

$A_{n \times n}$ — real

$$A^t = \begin{cases} A & \text{symmetric} \\ -A & \text{skew-sym} \\ A^{-1} & \text{orthogonal} \end{cases}$$

$A_{n \times n}$ — complex

$$\bar{A}^t = \begin{cases} A & \text{Hermitian} \\ -A & \text{skew-Hermitian} \\ A^{-1} & \text{unitary} \end{cases}$$

$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$, $\begin{bmatrix} 4 & 1-3i \\ 1+3i & 7 \end{bmatrix}$, $\begin{bmatrix} 3i & 2+i \\ -2+i & -i \end{bmatrix}$, $\begin{bmatrix} \frac{1}{2}i & \frac{1}{2}\sqrt{3} \\ \frac{1}{2}\sqrt{3} & \frac{1}{2}i \end{bmatrix}$

sym, skew-sym $a_{jj} = 0$, orthogonal, Her., skew-Her. $a_{jj} = i\beta$, unitary

$$A = \frac{1}{2}(A + A^t) + \frac{1}{2}(A - A^t), \qquad A = \frac{1}{2}(A + \bar{A}^t) + \frac{1}{2}(A - \bar{A}^t)$$

Properties

(1) A is sym/Hermitian $\Longrightarrow$ $\lambda(A)$ is real

(2) A is skew-sym/skew-Her. $\Longrightarrow$ $\lambda(A) = 0$ or $i\beta$ with $\beta \in R$.

(3) A is orthogonal/unitary $\Longrightarrow$ $|\lambda(A)| = 1$,

$|\det A| = 1$

inner product

$$\vec{a}\cdot\vec{b} = \begin{cases} \vec{a}^{\,t}\,\vec{b} & \text{real} \\ \overline{\vec{a}}^{\,t}\,\vec{b} & \end{cases}$$

- A is orthogonal/unitary $\Longrightarrow$ $\vec{a}_j \cdot \vec{a}_k = \delta_{jk} = \begin{cases} 1 & j=k \\ 0 & j\neq k \end{cases}$

$A = [\vec{a}_1, \cdots, \vec{a}_n]$ i.e., $\{\vec{a}_1, \cdots, \vec{a}_n\}$ — orthonormal system

- Let $\vec{u} = A\vec{a}$, $\vec{v} = A\vec{b}$

  A is orthogonal/unitary $\Longrightarrow$ $\vec{a}\cdot\vec{b} = \vec{u}\cdot\vec{v}$

p338 #4, 2, p351 #4, 14 (A-Her., B-skew Her $\Rightarrow \overline{(BA)}^t = -AB$) or 6

• $A=\begin{bmatrix}0&1\\-1&0\end{bmatrix}$ $\quad 0=\det\begin{bmatrix}-\lambda&1\\-1&-\lambda\end{bmatrix}=\lambda^2+1 \Longrightarrow \lambda^2=-1$ $\quad \lambda=\pm i$

• $A=\begin{bmatrix}3i&2+i\\-2+i&-i\end{bmatrix}$ $\quad \bar{A}^t=\begin{bmatrix}-3i&2-i\\-2-i&i\end{bmatrix}^t=\begin{bmatrix}-3i&-2-i\\2-i&i\end{bmatrix}=\overset{A}{\overset{\shortparallel}{-\begin{bmatrix}3i&2+i\\-2+i&-i\end{bmatrix}}}$

$$0=\det\begin{bmatrix}3i-\lambda&2+i\\-2+i&-i-\lambda\end{bmatrix}=(\lambda-3i)(\lambda+i)+(2+i)(2-i)$$

$$=\lambda^2-2i\lambda+3+4+1=\lambda^2-2i\lambda+8$$

$$\lambda_{1,2}=\frac{2i\pm\sqrt{-4-32}}{2}=i\pm3i=-2i,\ 4i$$

- $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ $\quad 0 = \det \begin{bmatrix} 1-\lambda & 2 \\ 2 & 4-\lambda \end{bmatrix} = (\lambda-1)(\lambda-4) - 4 = \lambda^2 - 5\lambda = \lambda(\lambda-5)$

$\underline{\lambda_1 = 0, \ \lambda_2 = 5}$ real

$i^2 = -1$

$\bar{A}^t = \begin{bmatrix} 4 & 1+3i \\ 1-3i & 7 \end{bmatrix}^t = \begin{bmatrix} 4 & 1-3i \\ 1+3i & 7 \end{bmatrix} = A$

$a \pm ib$

$(a-b)(a+b) = a^2 - b^2$

- $A = \begin{bmatrix} 4 & 1-3i \\ 1+3i & 7 \end{bmatrix}$ $\quad 0 = \det \begin{bmatrix} 4-\lambda & 1-3i \\ 1+3i & 7-\lambda \end{bmatrix} = (\lambda-4)(\lambda-7) - (1-3i)(1+3i)$

$= \lambda^2 - 11\lambda + 28 - [1+9]$

$= \lambda^2 - 11\lambda + 18 = (\lambda-2)(\lambda-9) = 0$

$\lambda_1 = 2, \ \lambda_2 = 9$ real

P338 #4

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \quad A^t = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}, \quad A^{-1} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$A^t = A^{-1}$ orthogonal

$$0 = \det(A - \lambda I) = \begin{vmatrix} \cos\theta - \lambda & -\sin\theta \\ \sin\theta & \cos\theta - \lambda \end{vmatrix} = (\cos\theta - \lambda)^2 + \sin^2\theta = \lambda^2 - 2\cos\theta\,\lambda + 1$$

$$\lambda = \frac{2\cos\theta \pm \sqrt{4\cos^2\theta - 4}}{2} = \cos\theta \pm \sqrt{-\sin^2\theta} = \cos\theta \pm i\,|\sin\theta|$$

$$|\lambda| = \sqrt{\cos^2\theta + |\sin\theta|^2} = 1$$