§8.4 Eigenbases. Diagonalization. Quadratic Forms.

$A_{n\times n}$: $\lambda_1, \lambda_2, \cdots, \lambda_n$; $\vec{x}_1, \cdots, \vec{x}_n$

(1) $\lambda_i \neq \lambda_j \implies \vec{x}_1, \cdots, \vec{x}_n$ l. indep. $\implies R^n = \text{span}\{\vec{x}_1, \cdots, \vec{x}_n\}$ eigenbases

(2) A is sym. $\implies$ A has an orthonomal bases of eigenvectors for R^n $\quad \|\vec{x}_1\| = \sqrt{2}$

$\begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}$ $\quad \lambda_1 = 8, \ \lambda_2 = 2 \quad \vec{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \vec{x}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $\quad \vec{y}_1 = \underline{\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}}, \ \vec{y}_2 = \underline{\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}}$ $\quad [1, 1]\begin{bmatrix} 1 \\ -1 \end{bmatrix} = 0$

$\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$ $\quad \lambda_1 = 5, \ \lambda_2 = \lambda_3 = -3 \quad \vec{x}_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \vec{x}_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \vec{x}_3 = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$ $\quad \begin{bmatrix} 1 & 2 & -1 \\ 0 & 5 & -2 \\ 0 & -6 & 4 \end{bmatrix}$

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$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ $\quad \lambda_1 = \lambda_2 = 0 \quad \vec{x}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Similarities of Matrices. Diagonalization.

$A\vec{x} = \vec{b}$ $\quad x_i = \frac{b_i}{\lambda_i}$

$\hat{A} \sim A$

- $\hat{A}_{n\times n}$ is similar to A $\iff$ $\exists$ nonsingular P s.t. $\boxed{\hat{A} = P^{-1}AP}$

similarity transformation

- $\hat{A} = P^{-1}AP \implies$ • $\lambda(\hat{A}) = \lambda(A)$

  • $\vec{x}$ is eigenvector of $A \Rightarrow \vec{y} = P^{-1}\vec{x}$ is eigenvector of $\hat{A}$

$A\vec{x} = \lambda\vec{x}$

$\hat{A}P^{-1}\vec{x} = P^{-1}APP^{-1}\vec{x} = P^{-1}A\vec{x} = \lambda P^{-1}\vec{x}$

- $A_{n\times n}$ has a eigenbases $\{\vec{x}_1, \cdots, \vec{x}_n\}$ corresponding to $\lambda_1, \cdots, \lambda_n$

$AX = X\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$

$$\implies X^{-1}AX = \Lambda = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \quad \text{with } X = [\vec{x}_1, \cdots, \vec{x}_n]$$

Diagonalize $\begin{bmatrix} -19 & 7 \\ -42 & 16 \end{bmatrix}$

$AX = [A\vec{x}_1, \cdots, A\vec{x}_n] = [\lambda_1\vec{x}_1, \cdots, \lambda_n\vec{x}_n]$

Quadratic Forms in the components $[x_1,\cdots,x_n]^t=\vec{x}$

$$Q=\underset{1\times n}{\vec{x}^{\,t}}\,\underset{n\times n}{A}\,\underset{n\times 1}{\vec{x}}=\sum_{j=1}^{n}\sum_{k=1}^{n}a_{jk}x_jx_k=\begin{array}{l}a_{11}x_1^2+\cdots+a_{1n}x_1x_n\\ \qquad\cdots\\ +a_{n1}x_nx_1+\cdots+a_{nn}x_n^2\end{array}$$

$\underset{n\times n}{A}$ − sym. $\Longrightarrow A=X\Lambda X^{-1}$ with $X=[\vec{x}_1,\cdots,\vec{x}_n]$, $X^{-1}=X^t$

$X^{-1}AX=\Lambda$

$\Lambda=\mathrm{diag}(\lambda_i)$ $\quad X^{-1}X=X^tX=I$

$$Q=\vec{x}^{\,t}A\vec{x}=\vec{x}^{\,t}X\Lambda X^t\vec{x}=\left(X^t\vec{x}\right)^t\Lambda\left(X^t\vec{x}\right)$$

$$=\vec{y}^{\,t}\Lambda\vec{y}=\underline{\sum_{i=1}^{n}\lambda_iy_i^2}$$ canonical form

$$Q=17x_1^2-30x_1x_2+17x_2^2=\vec{x}^{\,t}\begin{bmatrix}17&-15\\-15&17\end{bmatrix}\vec{x}=\vec{y}^{\,t}\begin{bmatrix}2&0\\0&32\end{bmatrix}\vec{y},\qquad \vec{y}=\frac{1}{\sqrt{2}}\begin{bmatrix}1&1\\-1&1\end{bmatrix}\vec{x}$$

$$=2y_1^2+32y_2^2$$

$$Q=128\Longrightarrow y_1^2/8^2+y_2^2/2^2=1$$

P346 #24 $Q(\vec{x}) = \vec{x}^t A \vec{x}$, A - sym $\vec{x}^t B \vec{x}$

- A is $\begin{cases} \text{pos. def} & Q(\vec{x}) > 0 \quad \forall \vec{x} \neq \vec{0} \\ \text{neg. def} & Q(\vec{x}) < 0 \quad \forall \vec{x} \neq \vec{0} \\ \text{indef.} & \exists \vec{x}, \vec{y} \text{ s.t. } Q(\vec{x}) > 0 \text{ and } Q(\vec{y}) < 0 \end{cases}$

$$\Longleftrightarrow \begin{cases} \lambda(A) > 0 \\ \lambda(A) < 0 \\ \exists\ \lambda(A) > 0 \text{ and } \lambda(A) < 0 \end{cases}$$

Proof $Q(\vec{x}) = \sum_i \lambda_i y_i^2$

P346 #25

- $Q(\vec{x}) = \vec{x}^t A \vec{x}$ is pos. def. $\iff a_{11} > 0, \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} > 0, \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} > 0, \cdots, \det A > 0$

#22 $Q(\vec{x}) = 4x_1^2 + 12x_1x_2 + 13x_2^2 = 16$. Show that $Q(\vec{x}) > 0 \ \forall \vec{x} \in R^2$.

#23 $Q(\vec{x}) = -11x_1^2 + 84x_1x_2 + 24x_2^2 = 156$. Show that $Q(\vec{x}) < 0 \ \forall \vec{x} \in R^2$.