MATH 527 PRACTICE PROBLEMS

1. Which of the following are vector spaces?
   - i) The set of all 3x3 matrices A such that $\det A = 0$.
   - ii) The set of all 2x2 matrices A such that $A \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} A$.
   - iii) The set of all symmetric 3x3 matrices.

   A. iii) only B. i) and ii) C. i) and iii) D. ii) and iii) E. i), ii), and iii)

2. Which of the sets of vectors are linearly independent?
   - i) $(0, 0, 1)$, $(0, 1, 1)$, $(0, 3, 2)$
   - ii) $(1, 2, 3)$, $(4, 5, 6)$, $(7, 8, 9)$
   - iii) $(0, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$

   A. i) B. ii) C. iii) D. i) and iii) E. None

3. The inverse of the matrix $\begin{pmatrix} 2 & -1 \\ 8 & -5 \end{pmatrix}$ is

   A. $\begin{pmatrix} 5 & -1 \\ 4 & -1 \end{pmatrix}$

   B. $\begin{pmatrix} \frac{5}{2} & 1 \\ 4 & 2 \end{pmatrix}$

   C. $\begin{pmatrix} \frac{5}{2} & -\frac{1}{2} \\ 4 & 1 \end{pmatrix}$

   D. $\begin{pmatrix} \frac{5}{2} & -\frac{1}{2} \\ 4 & -1 \end{pmatrix}$

   E. Matrix has no inverse.

4. Suppose that the system $Ax = b$, where A is an $n \times n$ matrix, has no solutions. Which of the following are true?
   - i) The homogeneous equation $Ax = 0$ has infinitely many solutions.
   - ii) The rank of A is less than n.
   - iii) A has no inverse.

   A. iii) only
   B. i) and ii)
   C. i) and iii)
   D. ii) and iii)
   E. i), ii), and iii)

5. The rank of the matrix $\begin{pmatrix} 0 & 1 & 2 & 0 \\ 0 & 2 & 4 & 0 \\ 0 & -3 & -6 & 0 \end{pmatrix}$ is

A. 0
B. 1
C. 2
D. 3
E. 4

6. The eigenvalues for the matrix $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 3 & 6 & 3 \end{pmatrix}$ are

A. 1, 2, 3
B. 1, 2, 0
C. 2, 3, 4
D. 1, 3, 0
E. 2, 4, 0

7. The eigenvalues of $\begin{pmatrix} 0 & -1 & 0 \\ -1 & 1 & -1 \\ 0 & -1 & 0 \end{pmatrix}$ are 2, 0, and -1. An eigenvector corresponding to -1 is

A. $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ B. $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

C. $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ D. $\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$

E. $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

8. One solution to $y' = \begin{pmatrix} 3 & 0 \\ 1 & 2 \end{pmatrix} y$ is $y = \begin{pmatrix} 0 \\ e^{2t} \end{pmatrix}$. Another linearly independent solution is

A. $\begin{pmatrix} e^{2t} \\ e^{3t} \end{pmatrix}$
B. $\begin{pmatrix} 0 \\ e^{2t} + e^{3t} \end{pmatrix}$
C. $\begin{pmatrix} e^{3t} \\ 0 \end{pmatrix}$
D. $\begin{pmatrix} e^{3t} \\ e^{2t} \end{pmatrix}$
E. $\begin{pmatrix} e^{3t} \\ e^{2t} + e^{3t} \end{pmatrix}$

9. For the system

$$\begin{aligned} y_1' &= y_1 + 3y_2 \\ y_2' &= 4y_1 + 2y_2 \end{aligned}$$

the origin is

A. an unstable node
B. a stable node
C. a saddle point
D. a stable spiral point
E. an unstable spiral point

10. For the system

$$\begin{aligned} y_1' &= 6y_1 + 9y_2 \\ y_2' &= y_1 + 6y_2 \end{aligned}$$

the origin is

A. an unstable node
B. a stable node
C. a saddle point
D. a stable spiral point
E. an unstable spiral point

11. For the system

$$dx/dt = \frac{3xy}{1+x^2+y^2} - \frac{1+x^2}{1+y^2}$$

$$dy/dt = x^2 - y^2,$$

the point $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is

A. an unstable node
B. a stable node
C. a saddle point
D. a stable spiral point
E. an unstable spiral point

12. For the system

$$dx/dt = y$$

$$dy/dt = \sin x,$$

the point $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is

A. an unstable node
B. a stable node
C. a saddle point
D. a stable spiral point
E. an unstable spiral point

13. Assume that a fundamental matrix for the equation $x' = Ax$ is

$$X(t) = \begin{pmatrix} e^{-3t} & e^{-t} \\ -e^{-3t} & e^{-t} \end{pmatrix}.$$

Then the general solution of $x' = Ax + \begin{pmatrix} 2e^{-t} \\ 2 \end{pmatrix}$ is

A. $x(t) = X(t)c + \begin{pmatrix} 2e^{-t} \\ 2 \end{pmatrix}$
B. $x(t) = X(t)c + \begin{pmatrix} e^{2t} - e^{3t} \\ e^{t} + 1 \end{pmatrix}$
C. $x(t) = X(t)c + \begin{pmatrix} e^{2t} - e^{3t} \\ e^{t} + t \end{pmatrix}$
D. $x(t) = X(t)c + \begin{pmatrix} \frac{e^{2t}}{2} - \frac{e^{3t}}{3} \\ e^{t} + t \end{pmatrix}$
E. None of the above.

14. Given the Laplace transform

$$\mathcal{L}(\frac{e^{-1/(4t)}}{\sqrt{t}}) = \frac{\sqrt{\pi}e^{-\sqrt{s}}}{\sqrt{s}},$$

then, $\mathcal{L}(\frac{e^{-1/(4t)}}{t^{3/2}}) =$

A. $2\sqrt{\pi}e^{-\sqrt{s}}$
B. $\frac{\sqrt{\pi}}{2s}e^{-\sqrt{s}}(1 + 1/\sqrt{s})$
C. $\frac{\sqrt{\pi}}{2s}e^{-\sqrt{s}}(1 + \sqrt{s})$
D. $2\sqrt{\pi}2se^{-\sqrt{s}}(1 + 1/\sqrt{s})$
E. $\frac{\sqrt{\pi}}{2s}e^{-\sqrt{s}}\sqrt{s}$

15. The inverse Laplace transform of $4/(s^3 + 4s)$ is

A. $1 + e^{2t}$
B. $1 + e^{2t} + e^{-2t}$
C. $t + e^{2t}$
D. $1 + \cos t$
E. $1 - \cos 2t$

16. Compute the inverse Laplace transform

$$\mathcal{L}^{-1}\left(\frac{e^{-s}}{s+2}\right) =$$

($u(t)$ is The Heaviside step function)

A. $u(t-1)e^{-2t}$
B. $u(t-2)e^{-t}$
C. $u(t-1)e^{2}e^{-2t}$
D. $u(t-1)e^{-1}e^{-t}$
E. $u(t-1)e^{-2}e^{-2t}$

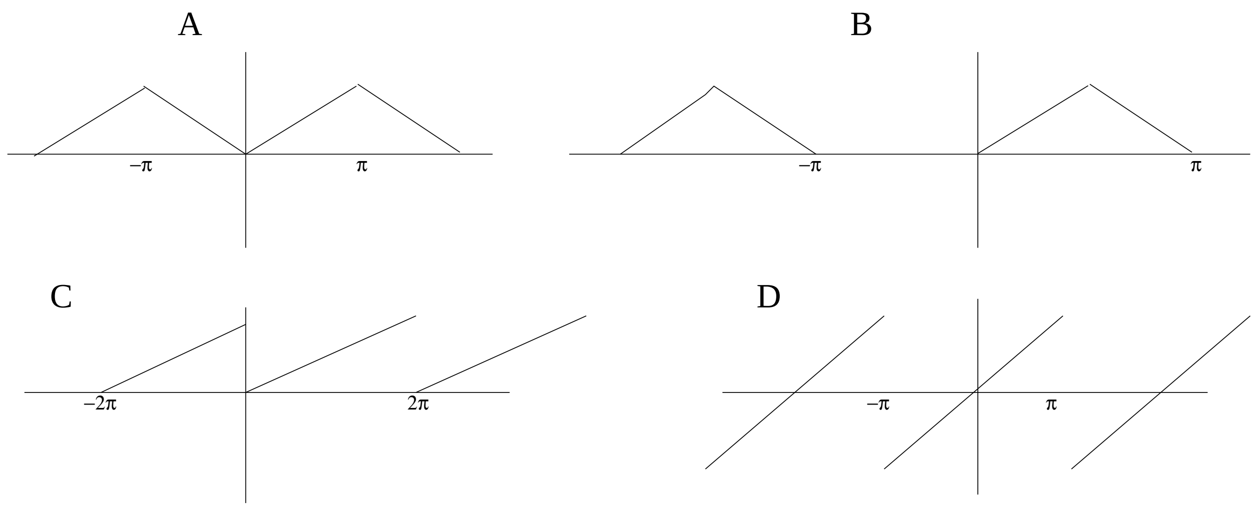
17. If $y' + y = u(t-1)e^{-2(t-1)} \quad y(0) = 1$, then $y(2) =$

A. e^{-1}
B. $2e^{-2} + e^{-1}$
C. $e^{-1} - 2e^{-2}$
D. $2e^{-1} + 2e^{-2}$
E. $2e^{-2}$

18. If $y'' + 2y' + y = \delta(t-1) \quad y(0) = y'(0) = 0$, then $y(2) =$

A. e^{-2}
B. e^{-1}
C. 1
D. e
E. e^{2}

In problems 18 and 19, match the given Fourier series with the portion of the graphs given below.



19. $\dfrac{1}{2} - \dfrac{1}{\pi}\displaystyle\sum_{n=1}^{\infty} \dfrac{1}{n} \sin nx$

A.
B.
C.
D.

20. $\dfrac{1}{2} - \dfrac{4}{\pi^2}\displaystyle\sum_{n=0}^{\infty} \dfrac{1}{(2n+1)^2} \cos(2n+1)x$

A.
B.
C.
D.

21. Given the fact that the Fourier cosine transform $\mathcal{F}_c(e^{-x}) = (\sqrt{2/\pi})/(1+w^2)$, the value of the integral

$$\frac{2}{\pi}\int_0^\infty \frac{\cos 2w}{1+w^2}dw$$

can be computed to be

A. $e^{-2}\cos 2$
B. $e^{-2}\sin 2$
C. e^{-2}
D. $-e^{-2}\cos 2$
E. $-e^{-2}\sin 2$

22. Given the fact that the (complex) Fourier transform $\mathcal{F}(e^{-x^2/2}) = e^{-w^2/2}$, then $\mathcal{F}(xe^{-x^2/2}) =$

A. $we^{-w^2/2}$
B. $-we^{-w^2/2}$
C. $iwe^{-w^2/2}$
D. $-iwe^{-w^2/2}$
E. $we^{-w^2/2} - 1$

23. Let

$$f(x) = \begin{cases} \sin 2x & 0 \le x \le \frac{\pi}{2} \\ 0 & \frac{\pi}{2} \le x \le \pi \end{cases}.$$

Then, $f(x)$ has the sine Fourier series

$$f(x) = \frac{1}{2}\sin 2x + \frac{4}{\pi}\sum_{n=0}^{\infty} \frac{(-1)^n}{4-(2n+1)^2}\sin(2n+1)x.$$

Using this information, if $u(x,t)$ satisfies the wave equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \qquad (c=1)$$
$$u(0,t) = u(\pi,t) = 0,$$
$$u(x,0) = f(x),$$
$$\frac{\partial u}{\partial t}(x,0) = 0,$$

then $u(\frac{\pi}{4}, \frac{\pi}{2}) =$

A. 0

B. $\frac{1}{2}$

C. $-\frac{1}{2}$

D. 1

E. -1

24. Let $u(x,t)$ satisfy the wave equation

$$\begin{aligned} &\frac{\partial^2 u}{\partial t^2} = 4\frac{\partial^2 u}{\partial x^2} && (c = 2) \\ &u(0,t) = u(\pi,t) = 0 && \\ &u(x,0) = \sin x && 0 \le x \le \pi \\ &\frac{\partial u}{\partial t}(x,0) = \sin 2x && 0 \le x \le \pi. \end{aligned}$$

Then $u(\frac{\pi}{4}, \frac{\pi}{8}) =$

A. $\frac{3}{4}$

B. $\frac{1}{4}$

C. $-\frac{1}{4}$

D. $\frac{1}{2}$

E. $-\frac{1}{2}$

25. Let $u(x,t)$ satisfy the heat equation

$$\frac{\partial u}{\partial t} = 4\frac{\partial^2 u}{\partial x^2} \qquad (c=2)$$
$$u(0,t) = u(\pi,t) = 0$$
$$u(x,0) = x(\pi - x) \qquad 0 < x < \pi.$$

Given that

$$x(\pi - x) = \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{1-(-1)^n}{n^3}\sin nx,$$

then $u(x,t) =$

A. $\frac{4}{\pi}\sum_{n=1}^{\infty} \frac{1-(-1)^n}{n^3}\cos nt \sin x$

B. $\frac{4}{\pi}\sum_{n=1}^{\infty} \frac{1-(-1)^n}{n^3}\sin 2nt \cos nx$

C. $\frac{4}{\pi}\sum_{n=1}^{\infty} \frac{1-(-1)^n}{n^3}\cos 2nt \sin 2nx$

D. $\frac{4}{\pi}\sum_{n=1}^{\infty} \frac{1-(-1)^n}{n^3}\sin nt \cos 2nx$

E. $\frac{4}{\pi}\sum_{n=1}^{\infty} \frac{1-(-1)^n}{n^3}e^{-4n^2 t}\sin nx$

26. If $u(x,t)$ satisfies the wave equation for an infinite string,

$$\begin{aligned}
&\frac{\partial^2 u}{\partial t^2} = 4\frac{\partial^2 u}{\partial x^2} && (c = 2) \\
&u(x,0) = \sin x && -\infty < x < \infty \\
&\frac{\partial u}{\partial t}(x,0) = \cos x && -\infty < x < \infty,
\end{aligned}$$

then, $u(0, \pi/4) =$

A. 0

B. $\frac{1}{4}$

C. $\frac{1}{2}$

D. $\frac{3}{4}$

E. 1

27. If $u(x,t)$ satisfies the heat equation in an infinite rod,

$$\frac{\partial u}{\partial t} = 4\frac{\partial^2 u}{\partial x^2} \qquad (c=2)$$
$$u(x,0) = \begin{cases} 1 & -1 < x < 1 \\ 0 & |x| > 1, \end{cases}$$

then $u(x,t) =$

A. $\frac{2}{\pi}\int_0^\infty \frac{\sin w}{w}(\cos wx)e^{-4w^2t}dw$
B. $\frac{2}{\pi}\int_0^\infty \frac{\cos w}{w}(\sin wx)e^{-4wt}dw$
C. $\frac{2}{\pi}\int_0^\infty \frac{\cos w}{w}(\cos wx)e^{-4w^2t}dw$
D. $\frac{2}{\pi}\int_0^\infty \frac{\sin w}{w}(\sin wx)e^{-4wt}dw$
E. $\frac{2}{\pi}\int_0^\infty \frac{\cos w}{w^2+1}(\cos wx)e^{-4wt}dw$

28. The solution of the 2-dimensional Laplace equation in polar coordinates

$$\nabla^2 u(r,\theta) = 0 \qquad (r<1)$$
$$u(1,\theta) = \cos 2\theta$$

is $u(r,\theta) =$

A. $\cos 2\theta$
B. $r\cos 2\theta$
C. $e^{r-1}\cos 2\theta$
D. $e^{2r-2}\cos 2\theta$
E. $r^2\cos 2\theta$

29. Let $u(x,t)$ be the solution to the 1-dimensional heat equation with insulated end conditions

$$\begin{aligned} &\frac{\partial u}{\partial t} = 4\frac{\partial^2 u}{\partial x^2} && (c=2) \\ &\frac{\partial u}{\partial x}(0,t) = 0, \quad \frac{\partial u}{\partial x}(\pi,t) = 0, && \\ &u(x,0) = \sin x && 0 \le x \le \pi. \end{aligned}$$

Given that

$$\sin x = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{4n^2-1}\cos 2nx \qquad 0 \le x \le \pi,$$

then $u(x,t) =$

A. $\frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{4n^2-1}(\cos 2nx)e^{-4(4n^2-1)t}$

B. $(\sin x)e^{-12t}$

C. $\frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{4n^2-1}(\sin 2nx)\cos 4nt$

D. $\frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{4n^2-1}(\sin 2nx)e^{-4(n^2-1)t}$

E. $\frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{4n^2-1}(\cos 2nx)e^{-16n^2t}$

30. The Fourier Series of $f(x) = x$ for $-\pi < x < \pi$ is $\displaystyle\sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \sin(nx)$. By Parseval's identity,

we can find the sum of the infinite series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} =$

A. $\dfrac{2\pi^2}{3}$

B. $\dfrac{7\pi^2}{12}$

C. $\dfrac{\pi^2}{6}$

D. $\dfrac{\pi^2}{12}$

E. $\dfrac{\pi^2}{2}$