MATH 527 PRACTICE PROBLEMS $\begin{pmatrix} 1&0&0\\0&0&0\\0&0&0 \end{pmatrix} + \begin{pmatrix} 0&0&0\\0&1&0\\0&0&1 \end{pmatrix} = \begin{pmatrix} 1&0&0\\0&1&0\\0&0&1 \end{pmatrix}$

1. Which of the following are vector spaces?
   i) The set of all 3x3 matrices A such that $\det A = 0$. No
   ii) The set of all 2x2 matrices A such that $A\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} A$. Yes
   iii) The set of all symmetric 3x3 matrices. Yes

   In ii) and iii) check addition and scalar multiplication

   A. iii) only B. i) and ii) C. i) and iii) D. ii) and iii) E. i), ii), and iii)

2. Which of the sets of vectors are linearly independent?
   i) $(0,0,1)$, $(0,1,1)$, $(0,3,2)$ No
   ii) $(1,2,3)$, $(4,5,6)$, $(7,8,9)$ No
   iii) $(0,0,0)$, $(0,1,0)$, $(0,0,1)$ No

   Each of 3x3 matrices is singular

   A. i) B. ii) C. iii) D. i) and iii) E. None

3. The inverse of the matrix $\begin{pmatrix} 2 & -1 \\ 8 & -5 \end{pmatrix}$ is

$$\left(\begin{array}{cc|cc} 2 & -1 & 1 & 0 \\ 8 & -5 & 0 & 1 \end{array}\right) \to \left(\begin{array}{cc|cc} 2 & -1 & 1 & 0 \\ 0 & -1 & -4 & 1 \end{array}\right) \to$$

$$\to \left(\begin{array}{cc|cc} 2 & 0 & 5 & -1 \\ 0 & -1 & -4 & 1 \end{array}\right) \to \left(\begin{array}{cc|cc} 1 & 0 & 5/2 & -1/2 \\ 0 & 1 & 4 & -1 \end{array}\right)$$

Alt. $\frac{1}{\det A}\begin{pmatrix} C_{11} & -C_{21} \\ -C_{12} & C_{22} \end{pmatrix} =$

$$= -\frac{1}{2}\begin{pmatrix} -5 & 1 \\ -8 & 2 \end{pmatrix} = \begin{pmatrix} 5/2 & -1/2 \\ 4 & -1 \end{pmatrix}$$

A. $\begin{pmatrix} 5 & -1 \\ 4 & -1 \end{pmatrix}$

B. $\begin{pmatrix} \frac{5}{2} & 1 \\ 4 & 2 \end{pmatrix}$

C. $\begin{pmatrix} \frac{5}{2} & -\frac{1}{2} \\ 4 & 1 \end{pmatrix}$

D. $\begin{pmatrix} \frac{5}{2} & -\frac{1}{2} \\ 4 & -1 \end{pmatrix}$

E. Matrix has no inverse.

4. Suppose that the system $Ax = b$, where A is an $n \times n$ matrix, has no solutions. Which of the following are true?
   i) The homogeneous equation $Ax = 0$ has infinitely many solutions. T
   ii) The rank of A is less than n. T
   iii) A has no inverse. T

   Since $Ax=b$ has no solutions, A is singular

   A. iii) only
   B. i) and ii)
   C. i) and iii)
   D. ii) and iii)
   E. i), ii), and iii)

5. The rank of the matrix $\begin{pmatrix} 0 & 1 & 2 & 0 \\ 0 & 2 & 4 & 0 \\ 0 & -3 & -6 & 0 \end{pmatrix}$ is

$$\downarrow$$

$$\begin{pmatrix} 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

A. 0
B. 1
C. 2
D. 3
E. 4

6. The eigenvalues for the matrix $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 3 & 6 & 3 \end{pmatrix}$ are

$$\det(A-\lambda I) = \det\begin{pmatrix} 1-\lambda & 0 & 1 \\ 0 & 2-\lambda & 0 \\ 3 & 6 & 3-\lambda \end{pmatrix} = -\lambda(\lambda-2)(\lambda-4)$$

<u>Alt.</u> A has two equal columns $\Rightarrow$ singular

$\Rightarrow \lambda = 0$ is an eigenvalue

$\lambda_1 + \lambda_2 + 0 = a_{11} + a_{22} + a_{33} = 6$

A. 1, 2, 3
B. 1, 2, 0
C. 2, 3, 4
D. 1, 3, 0
E. 2, 4, 0

7. The eigenvalues of $\begin{pmatrix} 0 & -1 & 0 \\ -1 & 1 & -1 \\ 0 & -1 & 0 \end{pmatrix}$ are 2, 0, and -1. An eigenvector corresponding to -1 is

$$A - (-1)I = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix} \Rightarrow$$

$$\Rightarrow \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

Null space spanned by $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

A. $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ B. $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

C. $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ D. $\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$

E. $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

Not an eigenvector

$y' = Ay$ has a solution $\vec{v} e^{\lambda t}$ where $\vec{v}$ is an eigenvector with the eigenvalue λ.

8. One solution to $y' = \begin{pmatrix} 3 & 0 \\ 1 & 2 \end{pmatrix} y$ is $y = \begin{pmatrix} 0 \\ e^{2t} \end{pmatrix}$. Another linearly independent solution is

General solution is

$c_1 e^{2t} \begin{pmatrix} 0 \\ 1 \end{pmatrix} + c_2 e^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

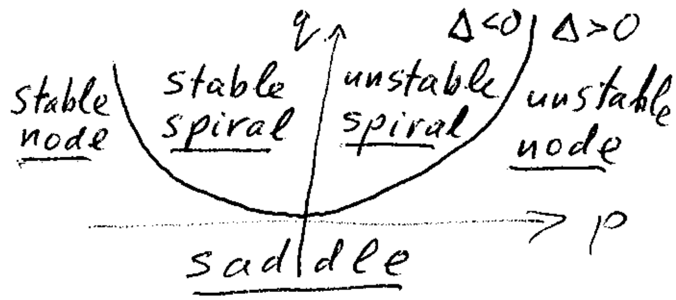
It is linearly independent of $\begin{pmatrix} 0 \\ e^{2t} \end{pmatrix}$ if $c_2 \neq 0$

A. $\begin{pmatrix} e^{2t} \\ e^{3t} \end{pmatrix}$
B. $\begin{pmatrix} 0 \\ e^{2t} + e^{3t} \end{pmatrix}$
C. $\begin{pmatrix} e^{3t} \\ 0 \end{pmatrix}$
D. $\begin{pmatrix} e^{3t} \\ e^{2t} \end{pmatrix}$
E. $\begin{pmatrix} e^{3t} \\ e^{2t} + e^{3t} \end{pmatrix}$

9. For the system

$A = \begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix}$

$$y_1' = y_1 + 3y_2$$
$$y_2' = 4y_1 + 2y_2$$



the origin is

$p = a_{11} + a_{22} = 3 > 0$
$q = \det A = -10 < 0$

A. an unstable node
B. a stable node
C. a saddle point
D. a stable spiral point
E. an unstable spiral point

10. For the system

$A = \begin{pmatrix} 6 & 9 \\ 1 & 6 \end{pmatrix}$

$$y_1' = 6y_1 + 9y_2$$
$$y_2' = y_1 + 6y_2$$

the origin is

$p = a_{11} + a_{22} = 12 > 0$
$q = \det A = 27 > 0$
$\Delta = p^2 - 4q = 144 - 108 > 0$

A. an unstable node
B. a stable node
C. a saddle point
D. a stable spiral point
E. an unstable spiral point

11. For the system

$$dx/dt = \frac{3xy}{1+x^2+y^2} - \frac{1+x^2}{1+y^2} = f(x,y)$$
$$dy/dt = x^2 - y^2, \quad = g(x,y)$$

the point $\begin{pmatrix}1\\1\end{pmatrix}$ is

$$f_x = \frac{3y(1+x^2+y^2) - 3xy \cdot 2x}{(1+x^2+y^2)^2} - \frac{2x}{1+y^2}$$
$$f_y = \frac{3x(1+x^2+y^2) - 3xy \cdot 2y}{(1+x^2+y^2)^2} + \frac{(1+x^2) \cdot 2y}{(1+y^2)^2}$$

A. an unstable node
B. a stable node
C. a saddle point
D. a stable spiral point
E. an unstable spiral point

$$g_x = 2x \quad , \quad g_y = -2y$$
$$\text{At } (x,y) = (1,1), \quad A = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix} = \begin{pmatrix} -\frac{2}{3} & \frac{5}{6} \\ 2 & -2 \end{pmatrix}$$
$$q = \det A = -\frac{1}{3} < 0$$

12. For the system

$$dx/dt = y \quad = f(x,y)$$
$$dy/dt = \sin x, \quad = g(x,y)$$

the point $\begin{pmatrix}0\\0\end{pmatrix}$ is

$$\text{At } (x,y) = (0,0),$$
$$A = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

A. an unstable node
B. a stable node
C. a saddle point
D. a stable spiral point
E. an unstable spiral point

$$q = \det A = -1 < 0$$

particular solution of $x' = Ax + g(t)$ $\quad x_p = X(t)u(t)$ where $u'(t) = X^{-1}(t)g(t)$

13. Assume that a fundamental matrix for the equation $x' = Ax$ is

$$X(t) = \begin{pmatrix} e^{-3t} & e^{-t} \\ -e^{-3t} & e^{-t} \end{pmatrix}. \qquad X^{-1}(t) = \begin{pmatrix} e^{3t}/2 & -e^{3t}/2 \\ e^{t}/2 & e^{t}/2 \end{pmatrix}$$

Then the general solution of $x' = Ax + \begin{pmatrix} 2e^{-t} \\ 2 \end{pmatrix}$ is

$$g(t) = \begin{pmatrix} 2e^{-t} \\ 2 \end{pmatrix}$$

$$u' = X^{-1}(t)g(t) = \begin{pmatrix} e^{2t} - e^{3t} \\ 1 + e^{t} \end{pmatrix}$$

$$u = \begin{pmatrix} e^{2t}/2 - e^{3t}/3 \\ t + e^{t} \end{pmatrix}$$

$$x_p = X(t)u(t) = \begin{pmatrix} e^{-t}/2 - 1/3 + te^{-t} + 1 \\ -e^{-t}/2 + 1/3 + te^{-t} + 1 \end{pmatrix}$$

A. $x(t) = X(t)c + \begin{pmatrix} 2e^{-t} \\ 2 \end{pmatrix}$

B. $x(t) = X(t)c + \begin{pmatrix} e^{2t} - e^{3t} \\ e^{t} + 1 \end{pmatrix}$

C. $x(t) = X(t)c + \begin{pmatrix} e^{2t} - e^{3t} \\ e^{t} + t \end{pmatrix}$

D. $x(t) = X(t)c + \begin{pmatrix} \frac{e^{2t}}{2} - \frac{e^{3t}}{3} \\ e^{t} + t \end{pmatrix}$

E. None of the above.

None of the answers has te^{-t}

14. Given the Laplace transform

$$\mathcal{L}\left(\frac{e^{-1/(4t)}}{\sqrt{t}}\right) = \frac{\sqrt{\pi}e^{-\sqrt{s}}}{\sqrt{s}},$$

then, $\mathcal{L}\left(\frac{e^{-1/(4t)}}{t^{3/2}}\right) = \mathcal{L}\left(\frac{e^{-1/4t}}{\sqrt{t}} \Big/ t\right)$.

A. $2\sqrt{\pi}e^{-\sqrt{s}}$

B. $\frac{\sqrt{\pi}}{2s}e^{-\sqrt{s}}(1 + 1/\sqrt{s})$

C. $\frac{\sqrt{\pi}}{2s}e^{-\sqrt{s}}(1 + \sqrt{s})$

D. $2\sqrt{\pi}2se^{-\sqrt{s}}(1 + 1/\sqrt{s})$

E. $\frac{\sqrt{\pi}}{2s}e^{-\sqrt{s}}\sqrt{s}$

$$\mathcal{L}(t f(t)) = -F'(s)$$

$$\mathcal{L}(f(t)/t) = \int_s^\infty F(\tau)\,d\tau$$

$$\int_s^\infty \frac{\sqrt{\pi}\,e^{-\sqrt{\tau}}}{\sqrt{\tau}}\,d\tau = -2\sqrt{\pi}e^{-\sqrt{\tau}}\Big|_s^\infty = 2\sqrt{\pi}e^{-\sqrt{s}}$$

$$\boxed{v = \sqrt{\tau}, \quad dv = \frac{d\tau}{2\sqrt{\tau}}}$$

15. The inverse Laplace transform of $4/(s^3+4s)$ is

$$\frac{4}{s(s^2+4)} = \frac{1}{s} - \frac{s}{s^2+4}$$

A. $1+e^{2t}$
B. $1+e^{2t}+e^{-2t}$
C. $t+e^{2t}$
D. $1+\cos t$
E. $1-\cos 2t$

16. Compute the inverse Laplace transform

$$\mathcal{L}^{-1}\left(\frac{e^{-s}}{s+2}\right) = \mathcal{L}^{-1}\left(e^{-s}F(s)\right)$$

($u(t)$ is The Heaviside step function)

s-shift $F(s)=\frac{1}{s+2}$, $f(t)=e^{-2t}$

t-shift $e^{-s}F(s)=\mathcal{L}\left(u(t-1)\,f(t-1)\right)$

$f(t-1)=e^{-2(t-1)}=e^{2}e^{-2t}$

A. $u(t-1)e^{-2t}$
B. $u(t-2)e^{-t}$
C. $u(t-1)e^{2}e^{-2t}$
D. $u(t-1)e^{-1}e^{-t}$
E. $u(t-1)e^{-2}e^{-2t}$

17. If $y'+y=u(t-1)e^{-2(t-1)}$ $y(0)=1$, then $y(2)=$

$$sY-1+Y=\frac{e^{-s}}{s+2}$$

$$Y=\frac{e^{-s}}{(s+1)(s+2)}+\frac{1}{s+1}=\frac{e^{-s}}{s+1}-\frac{e^{-s}}{s+2}+\frac{1}{s+1}$$

$$y(t)=u(t-1)e^{-(t-1)}-u(t-1)e^{-2(t-1)}+e^{-t},\quad y(2)=e^{-1}-e^{-2}+e^{-2}$$

A. e^{-1}
B. $2e^{-2}+e^{-1}$
C. $e^{-1}-2e^{-2}$
D. $2e^{-1}+2e^{-2}$
E. $2e^{-2}$

18. If $y''+2y'+y=\delta(t-1)$ $y(0)=y'(0)=0$, then $y(2)=$

$$s^2Y+2sY+Y=e^{-s}$$

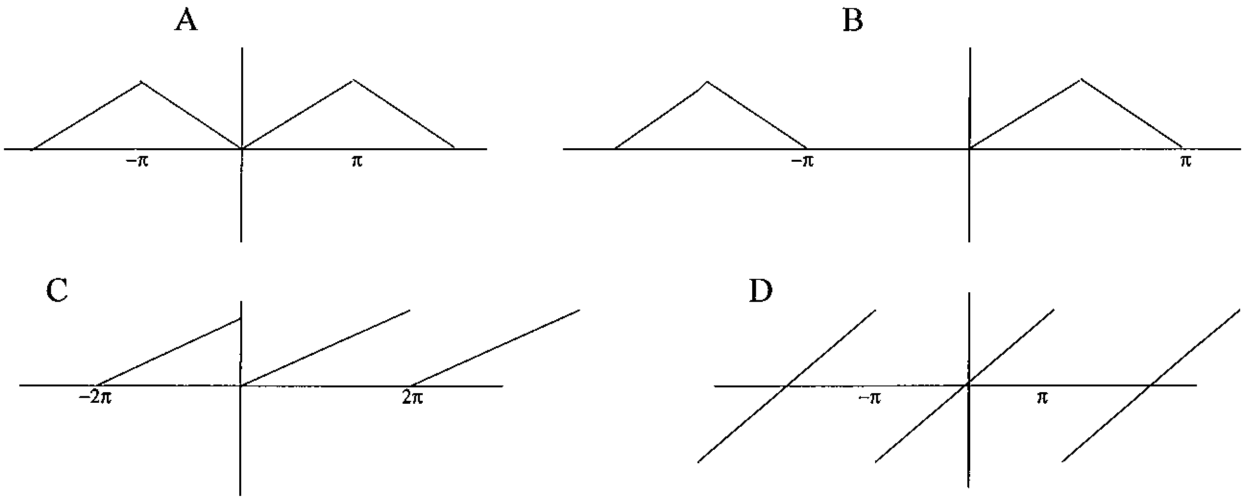
$$Y=\frac{e^{-s}}{(s+1)^2};\quad F(s)=\frac{1}{(s+1)^2},\ f(t)=te^{-t}$$

$$y(t)=u(t-1)\,f(t-1)=u(t-1)(t-1)e^{-(t-1)}$$

$$y(2)=e^{-1}$$

A. e^{-2}
B. e^{-1}
C. 1
D. e
E. e^2

In problems 19 and 20, match the given Fourier series with the portion of the graphs given below. use symmetry!



19. $\frac{1}{2} - \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin nx$

$f(x) - \frac{1}{2}$ is odd

A.
B.
C.
D.

20. $\frac{1}{2} - \frac{4}{\pi^2}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}\cos(2n+1)x$

$f(x)$ is even

A.
B.
C.
D.

Alt. Compute Fourier coefficients

$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx$

$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx, \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$

21. Given the fact that the Fourier cosine transform $\mathcal{F}_c(e^{-x}) = (\sqrt{2/\pi})/(1+w^2)$, the value of the integral

$$\frac{2}{\pi}\int_0^\infty \frac{\cos 2w}{1+w^2}dw$$

can be computed to be

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \mathcal{F}_c(w) \cos wx\, dw$$

At $x=2$,

$$e^{-2} = f(2) = \frac{2}{\pi}\int_0^\infty \frac{\cos 2w}{1+w^2}dw$$

A. $e^{-2}\cos 2$
B. $e^{-2}\sin 2$
C. e^{-2}
D. $-e^{-2}\cos 2$
E. $-e^{-2}\sin 2$

22. Given the fact that the (complex) Fourier transform $\mathcal{F}(e^{-x^2/2}) = e^{-w^2/2}$, then $\mathcal{F}(xe^{-x^2/2}) =$

$$\left(e^{-x^2/2}\right)' = -xe^{-x^2/2}$$

$$\mathcal{F}(f') = iw\mathcal{F}(f)$$

A. $we^{-w^2/2}$
B. $-we^{-w^2/2}$
C. $iwe^{-w^2/2}$
D. $-iwe^{-w^2/2}$
E. $we^{-w^2/2} - 1$

23. Let

$$f(x) = \begin{cases} \sin 2x & 0 \le x \le \frac{\pi}{2} \\ 0 & \frac{\pi}{2} \le x \le \pi \end{cases}.$$

Then, $f(x)$ has the sine Fourier series

$$f(x) = \frac{1}{2}\sin 2x + \frac{4}{\pi}\sum_{n=0}^{\infty} \frac{(-1)^n}{4-(2n+1)^2}\sin(2n+1)x.$$

Using this information, if $u(x,t)$ satisfies the wave equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \qquad (c = 1)$$
$$u(0,t) = u(\pi,t) = 0,$$
$$u(x,0) = f(x),$$
$$\frac{\partial u}{\partial t}(x,0) = 0,$$

then $u(\frac{\pi}{4}, \frac{\pi}{2}) =$

$u(x,t) = \sum_{k=1}^{\infty}\left(B_k \cos kt + B_k^* \sin kt\right)\sin kx$

where B_k are Fourier sine series coefficients of $f(x)$, and $B_k^* = 0$ since $\frac{\partial u}{\partial t}(x,0) = 0$.

A. 0

B. $\frac{1}{2}$

C. $-\frac{1}{2}$

D. 1

E. -1

At $t = \frac{\pi}{2}$, $\cos\left(\frac{k\pi}{2}\right) = 0$ if $k = 2n+1$ is odd.

Hence $u\left(\frac{\pi}{4}, \frac{\pi}{2}\right) = \frac{1}{2}\cos\pi \sin\frac{\pi}{2} = \underline{-\frac{1}{2}}$

24. Let $u(x,t)$ satisfy the wave equation

$$\frac{\partial^2 u}{\partial t^2} = 4\frac{\partial^2 u}{\partial x^2} \qquad (c=2)$$
$$u(0,t) = u(\pi,t) = 0$$
$$u(x,0) = \sin x \qquad 0 \le x \le \pi$$
$$\frac{\partial u}{\partial t}(x,0) = \sin 2x \qquad 0 \le x \le \pi.$$

Then $u(\frac{\pi}{4}, \frac{\pi}{8}) =$

A. $\frac{3}{4}$

B. $\frac{1}{4}$

C. $-\frac{1}{4}$

D. $\frac{1}{2}$

E. $-\frac{1}{2}$

$$u(x,t) = \sum_{n=1}^{\infty}\left(B_n \cos(2nt) + B_n^* \sin(2nt)\right)\sin(nx)$$

$$u(x,0) = \sum_{n=1}^{\infty} B_n \sin(nx) \Rightarrow B_1 = 1,$$
$$B_n = 0 \text{ for } n>1$$

$$\frac{\partial u}{\partial t}(x,0) = \sum_{n=1}^{\infty} 2n B_n^* \sin(nx) = \sin 2x$$
$$\Rightarrow B_1^* = 0, \quad B_2^* = \frac{1}{4}, \quad B_n^* = 0 \text{ for } n>2$$

$$u(x,t) = \cos(2t)\sin x + \frac{1}{4}\sin(4t)\sin(2x)$$
$$u\left(\frac{\pi}{4}, \frac{\pi}{8}\right) = \cos\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{4}\right) + \frac{1}{4}\sin\left(\frac{\pi}{2}\right)\sin\left(\frac{\pi}{2}\right) = \frac{3}{4}$$

Alt. One can use d'Alembert solution for odd 2π periodic extension of $u(x,t)$ to $-\infty < x < \infty$

25. Let $u(x,t)$ satisfy the heat equation

$$\frac{\partial u}{\partial t} = 4\frac{\partial^2 u}{\partial x^2} \qquad (c=2)$$
$$u(0,t) = u(\pi,t) = 0$$
$$u(x,0) = x(\pi - x) \qquad 0 < x < \pi.$$

Given that

$$x(\pi - x) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n^3}\sin nx,$$

then $u(x,t) =$

$u(x,t) = \sum_{n=1}^{\infty} B_n \sin(nx) e^{-4n^2 t}$
where B_n are Fourier
sine series coefficients
of $u(x,0) = x(\pi - x)$

A. $\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n^3}\cos nt \sin x$

B. $\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n^3}\sin 2nt \cos nx$

C. $\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n^3}\cos 2nt \sin 2nx$

D. $\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n^3}\sin nt \cos 2nx$

E. $\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n^3}e^{-4n^2 t}\sin nx$

26. If $u(x,t)$ satisfies the wave equation for an infinite string,

$$\frac{\partial^2 u}{\partial t^2} = 4\frac{\partial^2 u}{\partial x^2} \qquad (c=2)$$
$$u(x,0) = \sin x = f(x) \quad -\infty < x < \infty$$
$$\frac{\partial u}{\partial t}(x,0) = \cos x = g(x) \quad -\infty < x < \infty,$$

then, $u(0,\pi/4) =$

A. 0
B. $\frac{1}{4}$
C. $\frac{1}{2}$
D. $\frac{3}{4}$
E. 1

D'Alembert solution

$$u(x,t) = \frac{1}{2}\left(f(x+ct) + f(x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds$$

$$= \frac{1}{2}\left(\sin(x+2t) + \sin(x-2t)\right)$$
$$+ \frac{1}{4}\left(\sin(x+2t) - \sin(x-2t)\right).$$

At $(x,t) = (0, \frac{\pi}{4})$,

$$u(0,\tfrac{\pi}{4}) = \frac{1}{2}\left(\sin\frac{\pi}{2} + \sin(-\tfrac{\pi}{2})\right)$$
$$+ \frac{1}{4}\left(\sin\frac{\pi}{2} - \sin(-\tfrac{\pi}{2})\right) = \frac{1}{2}$$

27. If $u(x,t)$ satisfies the heat equation in an infinite rod,

$$\frac{\partial u}{\partial t} = 4\frac{\partial^2 u}{\partial x^2} \qquad (c=2)$$

$$u(x,0) = \begin{cases} 1 & -1 < x < 1 \\ 0 & |x| > 1, \end{cases}$$

then $u(x,t) =$

$u(x,t) = \int_0^\infty \Big(A(\omega)\cos\omega x$

$+ B(\omega)\sin\omega x\Big) e^{-4\omega^2 t}\, d\omega$

Since $u(x,0)$ is even, $B(\omega)=0$.

$A(\omega) = \frac{1}{\pi}\int_{-1}^{1} \cos \omega v \, dv$

$= \frac{1}{\pi} \frac{\sin \omega v}{\omega}\Big|_{-1}^{1} = \frac{2}{\pi}\frac{\sin\omega}{\omega}$

A. $\frac{2}{\pi}\int_0^\infty \frac{\sin w}{w}(\cos wx)e^{-4w^2 t}dw$

B. $\frac{2}{\pi}\int_0^\infty \frac{\cos w}{w}(\sin wx)e^{-4wt}dw$

C. $\frac{2}{\pi}\int_0^\infty \frac{\cos w}{w}(\cos wx)e^{-4w^2 t}dw$

D. $\frac{2}{\pi}\int_0^\infty \frac{\sin w}{w}(\sin wx)e^{-4wt}dw$

E. $\frac{2}{\pi}\int_0^\infty \frac{\cos w}{w^2+1}(\cos wx)e^{-4wt}dw$

28. The solution of the 2-dimensional Laplace equation in polar coordinates

$$\nabla^2 u(r,\theta) = 0 \qquad (r<1)$$

$$u(1,\theta) = \cos 2\theta$$

is $u(r,\theta) =$ $r^2 \cos 2\theta$

A. $\cos 2\theta$
B. $r\cos 2\theta$
C. $e^{r-1}\cos 2\theta$
D. $e^{2r-2}\cos 2\theta$
E. $r^2\cos 2\theta$

29. Let $u(x,t)$ be the solution to the 1-dimensional heat equation with insulated end conditions

$$\frac{\partial u}{\partial t} = 4\frac{\partial^2 u}{\partial x^2} \qquad (c=2)$$
$$\frac{\partial u}{\partial x}(0,t) = 0, \quad \frac{\partial u}{\partial x}(\pi,t) = 0,$$
$$u(x,0) = \sin x \qquad 0 \le x \le \pi.$$

Given that

$$\sin x = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{4n^2-1}\cos 2nx \qquad 0 \le x \le \pi,$$

then $u(x,t) =$

$$u(x,t) = \sum_{n=0}^{\infty} A_n \cos nx\, e^{-4n^2t}$$

where A_n are Fourier cosine series coefficients of $\sin x$, $0 \le x \le \pi$.

A. $\frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{4n^2-1}(\cos 2nx)e^{-4(4n^2-1)t}$

B. $(\sin x)e^{-12t}$

C. $\frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{4n^2-1}(\sin 2nx)\cos 4nt$

D. $\frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{4n^2-1}(\sin 2nx)e^{-4(n^2-1)t}$

E. $\frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{4n^2-1}(\cos 2nx)e^{-16n^2t}$

30. The Fourier Series of $f(x) = x$ for $-\pi < x < \pi$ is $\sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \sin(nx)$. By Parseval's identity,

we can find the sum of the infinite series $\sum_{n=1}^{\infty} \frac{1}{n^2} =$

$$\frac{1}{\pi}\|f\|^2 = 2a_0^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

$$\|f\|^2 = \int_{-\pi}^{\pi} x^2 dx = \frac{x^3}{3}\Big|_{-\pi}^{\pi} = \frac{2\pi^3}{3}$$

$$\frac{1}{\pi} \cdot \frac{2\pi^3}{3} = \sum_{n=1}^{\infty} \frac{4}{n^2}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

A. $\frac{2\pi^2}{3}$

B. $\frac{7\pi^2}{12}$

C. $\frac{\pi^2}{6}$

D. $\frac{\pi^2}{12}$

E. $\frac{\pi^2}{2}$