

Name: \_\_\_\_\_  
PUID#: \_\_\_\_\_

**Midterm 1A– Math 304 (9/29/09)**  
**SHOW ALL RELEVANT WORK!!!**

1. (15pts) Determine the radius of convergence of the given power series.

(1)  $\sum_{n=1}^{\infty} \frac{(3x-1)^n}{n^3}$ : (1)  $1/3$ , (2)  $1$ , (3)  $3$ , (4)  $\infty$ , or (5) none of the above.

**solution. (1)**

$$L = \lim \left| \frac{(3x-1)^{n+1}}{(n+1)^3} / \frac{(3x-1)^n}{n^3} \right| = |3x-1| \lim \frac{(n+1)^3}{n^3} = |3x-1| < 1,$$

which implies

$$\left| x - \frac{1}{3} \right| < \frac{1}{3}.$$

(2)  $\sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$ : (1)  $0$ , (2)  $2$ , (3)  $4$ , (4)  $\infty$ , or (5) none of the above.

**solution. (4)**

$$L = \lim \left| \frac{(-1)^{n+1} x^{2(n+1)}}{2^{2(n+1)}((n+1)!)^2} / \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2} \right| = \lim \left| \frac{x^2}{2^2(n+1)^2} \right| = 0 < 1,$$

which implies it converges for all  $x$ . That is, the radius of convergence is  $\infty$ .

2. (15pts) For the differential equation  $y'' - xy' - y = 0$  at  $x_0 = 0$ , (1) Find the recurrence relation, (2) find three terms in each of two linearly independent solutions (unless the series terminates sooner), and (3) if possible, find the general term in each solution.

**solution.** Let  $y = \sum_{n=0}^{\infty} a_n x^n$ , then

$$\begin{aligned} 0 &= \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=0}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n \\ &= \sum_{n=0}^{\infty} ((n+2)(n+1)a_{n+2} - (n+1)a_n) x^n. \end{aligned}$$

The recurrence relation is

$$a_{n+2} = \frac{a_n}{n+2} \quad \text{for } n = 0, 1, 2, \dots$$

which implies

$$\begin{aligned} a_2 &= \frac{a_0}{2}, & a_4 &= \frac{a_2}{4} = \frac{a_0}{4 \cdot 2}, & a_6 &= \frac{a_4}{6} = \frac{a_0}{6 \cdot 4 \cdot 2}, \dots, & a_{2k} &= \frac{a_0}{2 \cdot 4 \dots (2k)} = \frac{a_0}{2^k \cdot k!}, \\ a_3 &= \frac{a_1}{3}, & a_5 &= \frac{a_3}{5} = \frac{a_1}{5 \cdot 3}, & a_7 &= \frac{a_5}{7} = \frac{a_1}{7 \cdot 5 \cdot 3}, \dots, & a_{2k+1} &= \frac{a_1}{3 \cdot 5 \dots (2k+1)} = \frac{a_1 2^k \cdot k!}{(2k+1)!}. \end{aligned}$$

Now, two linearly independent solutions are

$$y_1 = \sum_{k=0}^{\infty} a_{2k} x^{2k} = a_0 \sum_{k=0}^{\infty} \frac{1}{2^k \cdot k!} x^{2k} \quad \text{and} \quad y_2 = \sum_{k=0}^{\infty} a_{2k+1} x^{2k+1} = a_1 \sum_{k=0}^{\infty} \frac{2^k \cdot k!}{(2k+1)!} x^{2k+1}.$$

3. (15pts) Determine the lower bounds for the radii of convergence of series solutions at  $x_0 = 0$  and  $x_0 = 2$  for

$$(x^2 + x - 2)y'' + (x + 1)y' + 2y = 0.$$

**Solution.** Set  $0 = x^2 + x - 2 = (x + 2)(x - 1)$ . Then  $x = -2$  and  $1$ . At  $x_0 = 2$ ,

$$\text{dist}(-2, 2) = 4, \quad \text{dist}(1, 2) = 1 \quad \Rightarrow \rho \geq 1.$$

At  $x_0 = 0$ ,

$$\text{dist}(-2, 0) = 2, \quad \text{dist}(1, 0) = 1 \quad \Rightarrow \rho \geq 1.$$

4. (15pts) Let  $y(x)$  be the solution of the following Euler equation:

$$x^2 y'' + 3xy' + 5y = 0, \quad y(1) = 1, \quad y'(1) = -1.$$

The value  $y(2)$  is

- (1)  $(\cos(\ln 4) - \sin(\ln 4))/2$ , (2)  $(\cos(\ln 4) + \sin(\ln 4))/2$ , (3)  $\frac{1}{2} \cos(\ln 4)$ ,  
(4)  $\frac{1}{2} \sin(\ln 4)$ , or (5) none of the above.

**Solution. (3)** The characteristic equation is

$$0 = r(r - 1) + 3r + 5 = r^2 + 2r + 5 = (r + 1)^2 + 4 \Rightarrow r = -1 \pm 2i,$$

which gives the general solution

$$\begin{aligned} y(x) &= x^{-1} (c_1 \cos(2 \ln x) + c_2 \sin(2 \ln x)) \\ y' &= -x^{-2} ((c_1 - 2c_2) \cos(2 \ln x) + (c_2 + 2c_1) \sin(2 \ln x)). \end{aligned}$$

The initial conditions imply

$$1 = y(1) = c_1 \quad \text{and} \quad -1 = y'(1) = -c_1 + 2c_2 \Rightarrow c_1 = 1 \quad \text{and} \quad c_2 = 0.$$

Hence, the solution is

$$y(x) = x^{-1} \cos(2 \ln x)$$

and  $y(2) = (\cos(\ln 4))/2$ .

5. (15pts) Let  $S_r$  and  $S_i$  denote the sets of regular and irregular singular points of the given differential equations

$$x(1-x^2)^3 y'' + (1-x^2)^2 y' + 2(1+x)y = 0,$$

respectively. Then

- (1)  $S_r = \{-1, 0\}$  and  $S_i = \{1\}$
- (2)  $S_r = \{-1, 1\}$  and  $S_i = \{0\}$
- (3)  $S_r = \{0, 1\}$  and  $S_i = \{-1\}$
- (4)  $S_r = \{-1, 0, 1\}$  and  $S_i = \emptyset$
- (5)  $S_r = \{0\}$  and  $S_i = \{1\}$ .

**solution.** Let  $x(1-x^2)^3 = 0$ . Then  $x = -1, 0, 1$  are singular points. Since at  $x = -1$

$$\lim_{x \rightarrow -1} (x+1) \frac{(1-x^2)^2}{x(1-x^2)^3} = \lim_{x \rightarrow -1} \frac{1}{x(1-x)} = -\frac{1}{2}, \quad \lim_{x \rightarrow -1} (x+1)^2 \frac{2(1+x)}{x(1-x^2)^3} = \lim_{x \rightarrow -1} \frac{2}{x(1-x)^3} = -\frac{1}{4},$$

at  $x = 0$

$$\lim_{x \rightarrow 0} x \frac{(1-x^2)^2}{x(1-x^2)^3} = \lim_{x \rightarrow 0} \frac{1}{1-x^2} = 1 \quad \text{and} \quad \lim_{x \rightarrow 0} x^2 \frac{2(1+x)}{x(1-x^2)^3} = \lim_{x \rightarrow 0} \frac{2x(1+x)}{(1-x^2)^3} = 0$$

and at  $x = 1$

$$\lim_{x \rightarrow 1} (x-1) \frac{(1-x^2)^2}{x(1-x^2)^3} = \lim_{x \rightarrow 1} \frac{-1}{x(x+1)} = -2,$$

$$\lim_{x \rightarrow 1} (x-1)^2 \frac{2(1+x)}{x(1-x^2)^3} = \lim_{x \rightarrow 1} \frac{2}{x(1+x)(1-x^2)} \text{ DNE.}$$

Hence,  $x = -1, 0$  are regular singular points and  $x = 1$  is an irregular singular point. Hence,

$$S_r = \{-1, 0\} \text{ and } S_i = \{1\}.$$

6. (12pts) Determine whether the following set of vectors

$$\mathbf{x}^{(1)} = \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix}, \quad \mathbf{x}^{(2)} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad \text{and} \quad \mathbf{x}^{(3)} = \begin{pmatrix} -1 \\ -2 \\ 2 \end{pmatrix}$$

are linearly independent. If they are linearly dependent, find a linear relation among them.

**solution.** Since

$$\begin{pmatrix} 1 & 0 & -1 \\ 3 & 1 & -2 \\ -1 & 1 & 2 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \implies \begin{cases} c_2 = -c_3 \\ c_1 = c_3 \end{cases}$$

Hence, they are linearly dependent and satisfy the following linear relation:

$$\mathbf{x}^{(1)} - \mathbf{x}^{(2)} + \mathbf{x}^{(3)} = \mathbf{0}.$$

7. (13pts) Find all eigenvalues and eigenvectors of the given matrix  $A = \begin{pmatrix} 3 & -2 \\ 4 & 1 \end{pmatrix}$ .

**solution.** Set

$$\det(A - \lambda I) = \begin{vmatrix} 3 - \lambda & -2 \\ 4 & 1 - \lambda \end{vmatrix} = (3 - \lambda)(1 - \lambda) + 8 = \lambda^2 - 4\lambda + 11 = (\lambda - 2)^2 + 7 = 0$$

gives  $\lambda_1 = 2 + i\sqrt{7}$  and  $\lambda_2 = 2 - i\sqrt{7}$ . For  $\lambda_1 = 2 + i\sqrt{7}$ , the corresponding eigenvector  $\boldsymbol{\xi}^{(1)}$  is computed as follows

$$\begin{pmatrix} 1 - i\sqrt{7} & -2 \\ 4 & -1 - i\sqrt{7} \end{pmatrix} \implies \begin{pmatrix} 1 - i\sqrt{7} & -2 \\ 0 & 0 \end{pmatrix} \implies \xi_2 = \left(\frac{1}{2} - i\frac{\sqrt{7}}{2}\right)\xi_1.$$

Hence,  $\boldsymbol{\xi}^{(1)} = \begin{pmatrix} 2 \\ 1 - i\sqrt{7} \end{pmatrix}$ . For  $\lambda_2 = 2 - i\sqrt{7}$ , the corresponding eigenvector  $\boldsymbol{\xi}^{(2)}$  is

$$\boldsymbol{\xi}^{(2)} = \overline{\boldsymbol{\xi}^{(1)}} = \begin{pmatrix} 2 \\ 1 + i\sqrt{7} \end{pmatrix}.$$