Name: ____________________

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Midterm 1B– Math 304 (9/29/09)
SHOW ALL RELEVANT WORK!!!

1. (15pts) Determine the radius of convergence of the given power series.

(1) $\sum_{n=1}^{\infty} \frac{(x-1)^n}{2^n}$: (1) 1/2, (2) 1, (3) 2, (4) ∞, or (5) none of the above.

solution. (3)

$$L = \lim |\frac{(x-1)^{n+1}}{2^{n+1}} / \frac{(x-1)^n}{2^n}| = |x-1|/2 < 1,$$

which implies

$$|x-1| < 2.$$

(2) $\sum_{n=1}^{\infty} \frac{(-1)^n(3x-1)^{2n}}{2^{2n}}$: (1) 1/4, (2) 4/9, (3) 2/3, (4) 3/2, or (5) 2.

solution. (3)

$$L = \lim |\frac{(-1)^{n+1}(3x-1)^{2(n+1)}}{2^{2(n+1)}} / \frac{(-1)^n(3x-1)^{2n}}{2^{2n}}| = (3x-1)^2/4 < 1,$$

which implies

$$|x - \frac{1}{3}| < \sqrt{4/9} = 2/3.$$

2. (15pts) For the differential equation $y'' - xy' - y = 0$ at $x_0 = 0$, (1) Find the recurrence relation, (2) find three terms in each of two linearly independent solutions (unless the series terminates sooner), and (3) if possible, find the general term in each solution.

solution. Let $y = \sum_{n=0}^{\infty} a_n x^n$, then

$$\begin{aligned} 0 &= \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=0}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n \\ &= \sum_{n=0}^{\infty} \left((n+2)(n+1)a_{n+2} - (n+1)a_n\right) x^n. \end{aligned}$$

The recurrence relation is

$$a_{n+2} = \frac{a_n}{n+2} \quad \text{for} \quad n = 0, 1, 2, \cdots$$

which implies

$$\begin{aligned} &a_2 = \tfrac{a_0}{2}, \quad a_4 = \tfrac{a_2}{4} = \tfrac{a_0}{4\cdot 2}, \quad a_6 = \tfrac{a_4}{6} = \tfrac{a_0}{6\cdot 4\cdot 2}, \cdots, \quad a_{2k} = \tfrac{a_0}{2\cdot 4\cdots(2k)} = \tfrac{a_0}{2^k\cdot k!}, \\ &a_3 = \tfrac{a_1}{3}, \quad a_5 = \tfrac{a_3}{5} = \tfrac{a_1}{5\cdot 3}, \quad a_7 = \tfrac{a_5}{7} = \tfrac{a_1}{7\cdot 5\cdot 3}, \cdots, \quad a_{2k+1} = \tfrac{a_1}{3\cdot 5\cdots(2k+1)} = \tfrac{a_1 2^k\cdot k!}{(2k+1)!}. \end{aligned}$$

Now, two linearly independent solutions are

$$y_1 = \sum_{k=0}^{\infty} a_{2k} x^{2k} = a_0 \sum_{k=0}^{\infty} \frac{1}{2^k \cdot k!} x^{2k} \quad \text{and} \quad y_2 = \sum_{k=0}^{\infty} a_{2k+1} x^{2k+1} = a_1 \sum_{k=0}^{\infty} \frac{2^k \cdot k!}{(2k+1)!} x^{2k+1}.$$

3. (15pts) Determine the lower bounds for the radii of convergence of series solutions at $x_0 = 0$ and $x_0 = 2$ for

$$(x^2 + x - 2)y'' + (x+1)y' + 2y = 0.$$

Solution. Set $0 = x^2 + x - 2 = (x+2)(x-1)$. Then $x = -2$ and 1. At $x_0 = 2$,

$$dist(-2, 2) = 4, \quad dist(1, 2) = 1 \quad \Rightarrow \rho \geq 1.$$

At $x_0 = 0$,

$$dist(-2, 0) = 2, \quad dist(1, 0) = 1 \quad \Rightarrow \rho \geq 1.$$

4. (15pts) Let $y(x)$ be the solution of the following Euler equation:

$$x^2y'' + 3xy' + 5y = 0, \quad y(1) = 1, \quad y'(1) = -1.$$

The value $y(e)$ is

(1) $(\cos 2{+}\sin 2)/e$, (2) $\dfrac{1}{e}\cos 2$, (3) $(\cos 2{-}\sin 2)/e$, (4) $\dfrac{1}{e}\sin 2$, or (5) none of the above.

Solution. (2) The characteristic equation is

$$0 = r(r-1) + 3r + 5 = r^2 + 2r + 5 = (r+1)^2 + 4 \quad \Rightarrow \quad r = -1 \pm 2i,$$

which gives the general solution

$$\begin{aligned} y(x) &= x^{-1}\left(c_1 \cos(2\ln x) + c_2 \sin(2\ln x)\right) \\ y' &= -x^{-2}\left((c_1 - 2c_2)\cos(2\ln x) + (c_2 + 2c_1)\sin(2\ln x)\right). \end{aligned}$$

The initial conditions imply

$$1 = y(1) = c_1 \quad \text{and} \quad -1 = y'(1) = -c_1 + 2c_2 \quad \Rightarrow \quad c_1 = 1 \text{ and } c_2 = 0.$$

Hence, the solution is

$$y(x) = x^{-1}\cos(2\ln x)$$

and $y(e) = (\cos 2)/e$.

5. (15pts) Let S_r and S_i denote the sets of regular and irregular singular points of the given differential equations

$$x^2(1-x)y'' + (x-2)y' - 3xy = 0,$$

respectively. Then

(1) $S_r = \{1, 2\}$ and $S_i = \{0\}$

(2) $S_r = \{1\}$ and $S_i = \{0\}$

(3) $S_r = \{0, 1\}$ and $S_i = \emptyset$

(4) $S_r = \{0, 1, 2\}$ and $S_i = \emptyset$

(5) $S_r = \{0\}$ and $S_i = \{1\}$.

solution. Let $x^2(1-x) = 0$. Then $x = 0, 1$ are singular points. Since at $x = 0$

$$\lim_{x\to 0} x\frac{x-2}{x^2(1-x)} = \lim_{x\to 0}\frac{x-2}{x(1-x)} \text{ DNE} \quad \text{and} \quad \lim_{x\to 0} x^2\frac{-3x}{x^2(1-x)} = \lim_{x\to 0}\frac{-3x}{1-x} = 0$$

and at $x = 1$

$$\lim_{x\to 1}(x-1)\frac{x-2}{x^2(1-x)} = \lim_{x\to 1}\frac{2-x}{x^2} = 1 \quad \text{and} \quad \lim_{x\to 1}(x-1)^2\frac{-3x}{x^2(1-x)} = \lim_{x\to 1}\frac{-3(1-x)}{x} = 0$$

$x = 0$ is a irregular singular point and $x = 1$ is a regular singular point. Hence,

$$S_r = \{1\} \text{ and } S_i = \{0\}.$$

6. (12pts) Determine whether the following set of vectors

$$\mathbf{x}^{(1)} = \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix}, \quad \mathbf{x}^{(2)} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad \text{and} \quad \mathbf{x}^{(3)} = \begin{pmatrix} -1 \\ -2 \\ 2 \end{pmatrix}$$

are linearly independent. If they are linearly dependent, find a linear relation among them.
solution. Since

$$\begin{pmatrix} 1 & 0 & -1 \\ 3 & 1 & -2 \\ -1 & 1 & 2 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \Longrightarrow \begin{cases} c_2 = -c_3 \\ c_1 = c_3 \end{cases}$$

Hence, they are linearly dependent and satisfy the following linear relation:

$$\mathbf{x}^{(1)} - \mathbf{x}^{(2)} + \mathbf{x}^{(3)} = 0.$$

7. (13pts) Find all eigenvalues and eigenvectors of the given matrix $A = \begin{pmatrix} 3 & -2 \\ 4 & 1 \end{pmatrix}$.
solution. Set

$$det(A - \lambda I) = \begin{vmatrix} 3-\lambda & -2 \\ 4 & 1-\lambda \end{vmatrix} = (3-\lambda)(1-\lambda) + 8 = \lambda^2 - 4\lambda + 11 = (\lambda - 2)^2 + 7 = 0$$

gives $\lambda_1 = 2 + i\sqrt{7}$ and $\lambda_2 = 2 - i\sqrt{7}$. For $\lambda_1 = 2 + i\sqrt{7}$, the corresponding eigenvector $\boldsymbol{\xi}^{(1)}$ is computed as follows

$$\begin{pmatrix} 1 - i\sqrt{7} & -2 \\ 4 & -1 - i\sqrt{7} \end{pmatrix} \Longrightarrow \begin{pmatrix} 1 - i\sqrt{7} & -2 \\ 0 & 0 \end{pmatrix} \Longrightarrow \xi_2 = (\frac{1}{2} - i\frac{\sqrt{7}}{2})\xi_1.$$

Hence, $\boldsymbol{\xi}^{(1)} = \begin{pmatrix} 2 \\ 1 - i\sqrt{7} \end{pmatrix}$. For $\lambda_2 = 2 - i\sqrt{7}$, the corresponding eigenvector $\boldsymbol{\xi}^{(2)}$ is

$$\boldsymbol{\xi}^{(2)} = \overline{\boldsymbol{\xi}^{(1)}} = \begin{pmatrix} 2 \\ 1 + i\sqrt{7} \end{pmatrix}.$$