Name: ____________________
PUID#: ____________________

Midterm 2– Math 304 (10/27/09)
SHOW ALL RELEVANT WORK!!!

1. (20pts) For the given initial value problem

$$\mathbf{x}'(t) = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} \mathbf{x}(t) \quad \text{and} \quad \mathbf{x}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

eigenvalues of the coefficient matrix are $\lambda_1 = -3$ and $\lambda_2 = 2$,

- Compute the corresponding eigenvectors,
- Find the general solution of the system,
- Find the solution of the initial value problem;
- Describe the behavior of the solution as $t \to \infty$.

solution. Set

$$det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 1 \\ 4 & -2-\lambda \end{vmatrix} = (\lambda - 1)(\lambda + 2) - 4 = \lambda^2 + \lambda - 6 = (\lambda - 2)(\lambda + 3) = 0$$

gives $\lambda_1 = -3$ and $\lambda_2 = 2$. For $\lambda_1 = -3$, the corresponding eigenvector $\boldsymbol{\xi}^{(1)}$ is computed as follows

$$\begin{pmatrix} 4 & 1 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Longrightarrow \begin{pmatrix} 4 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Longrightarrow 4\xi_1 = -\xi_2 \Longrightarrow \boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ -4 \end{pmatrix}.$$

For $\lambda_2 = 2$, the corresponding eigenvector $\boldsymbol{\xi}^{(2)}$ is

$$\begin{pmatrix} -1 & 1 \\ 4 & -4 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Longrightarrow \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Longrightarrow \xi_1 = \xi_2 \Longrightarrow \boldsymbol{\xi}^{(2)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

The general solution is

$$\mathbf{x}(t) = c_1 e^{-3t} \begin{pmatrix} 1 \\ -4 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

At the initial condition,

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \mathbf{x}(0) = c_1 \begin{pmatrix} 1 \\ -4 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -4 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

The solution of the initial value problem is

$$\mathbf{x}(t) = e^{2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

As $t \to \infty$, the solution approaches ∞ along the direction $(1,\, 1)^t$.

2. (15pts) For the given system of equations

$$\mathbf{x}'(t) = \begin{pmatrix} -1 & -1 \\ 2 & -1 \end{pmatrix} \mathbf{x}(t),$$

eigenvalues of the coefficient matrix are $\lambda_1 = -1 + i\sqrt{2}$ and $\lambda_2 = -1 - i\sqrt{2}$,

- Compute the corresponding eigenvectors,
- Find the general solution of the system,
- Describe the behavior of the solution as $t \to \infty$.

solution. Set

$$det(A - \lambda I) = \begin{vmatrix} -1-\lambda & -1 \\ 2 & -1-\lambda \end{vmatrix} = (\lambda+1)^2 + 2 = 0$$

gives $\lambda_1 = -1 + i\sqrt{2}$ and $\lambda_2 = -1 - i\sqrt{2}$. For $\lambda_1 = -1 + i\sqrt{2}$, the corresponding eigenvector $\boldsymbol{\xi}^{(1)}$ is computed as follows

$$\begin{pmatrix} -i\sqrt{2} & -1 \\ 2 & -i\sqrt{2} \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Longrightarrow \begin{pmatrix} -i\sqrt{2} & -1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Longrightarrow \xi_2 = -i\sqrt{2}\xi_1 \Longrightarrow \boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ -i\sqrt{2} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + i \begin{pmatrix} 0 \\ -\sqrt{2} \end{pmatrix}.$$

For $\lambda_2 = -1 - i\sqrt{2}$, the corresponding eigenvector $\boldsymbol{\xi}^{(2)}$ is

$$\boldsymbol{\xi}^{(2)} = \overline{\boldsymbol{\xi}^{(1)}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + i \begin{pmatrix} 0 \\ \sqrt{2} \end{pmatrix}$$

The general solution is

$$\mathbf{x}(t) = e^{-t} \left(c_1 \begin{pmatrix} \cos(\sqrt{2}t) \\ \sqrt{2}\sin(\sqrt{2}t) \end{pmatrix} + c_2 \begin{pmatrix} \sin(\sqrt{2}t) \\ -\sqrt{2}\cos(\sqrt{2}t) \end{pmatrix} \right).$$

As $t \to \infty$, the solution spirals to $(0, 0)^t$.

3. (15pts) For the given system of equations

$$\mathbf{x}'(t) = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix} \mathbf{x}(t)$$

- Compute the eigenvalues and eigenvectors of the coefficient matrix;
- Find the general solution of the system.

solution. Set

$$det(A-\lambda I) = \begin{vmatrix} 1-\lambda & -1 \\ 1 & 3-\lambda \end{vmatrix} = (\lambda-1)(\lambda-3)+1 = \lambda^2-4\lambda+4 = (\lambda-2)^2 = 0$$

gives $\lambda_1 = 2$ and $\lambda_2 = 2$. For $\lambda_1 = 2$, the corresponding eigenvector $\boldsymbol{\xi}$ is computed as follows

$$\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Longrightarrow \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Longrightarrow \xi_1+\xi_2 = 0$$

$$\Longrightarrow \boldsymbol{\xi} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \Longrightarrow \mathbf{x}^{(1)}(t) = e^{2t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

To find the second solution, we have

$$\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \Longrightarrow \eta_1+\eta_2 = -1 \Longrightarrow \boldsymbol{\eta} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

$$\Longrightarrow \mathbf{x}^{(2)}(t) = te^{2t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + e^{2t}\begin{pmatrix} 0 \\ -1 \end{pmatrix}.$$

The general solution is

$$\mathbf{x}(t) = c_1e^{2t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2\left[te^{2t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + e^{2t}\begin{pmatrix} 0 \\ -1 \end{pmatrix}\right].$$

4. (10pts) For the given system of equations

$$\mathbf{x}'(t) = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}\mathbf{x}(t),$$

eigenvalues of the coefficient matrix are $\lambda_1 = -1$, $\lambda_2 = -1$, and $\lambda_3 = 2$,

- Compute the corresponding eigenvectors of the coefficient matrix,
- Find the general solution of the system.

solution. For $\lambda_1 = -1$ and $\lambda_2 = -1$, the corresponding eigenvectors $\boldsymbol{\xi}^{(1)}$ and $\boldsymbol{\xi}^{(2)}$ are computed as follows

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Longrightarrow \xi_1 + \xi_2 + \xi_3 = 0 \Longrightarrow \boldsymbol{\xi}^{(1)} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \text{ and } \boldsymbol{\xi}^{(2)} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}.$$

For $\lambda_3 = 2$, the corresponding eigenvector $\boldsymbol{\xi}^{(3)}$ is

$$\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \Longrightarrow \begin{pmatrix} -2 & 1 & 1 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Longrightarrow \begin{cases} -2\xi_1 + \xi_2 + \xi_3 = 0 \\ \xi_1 - \xi_2 = 0 \end{cases} \Longrightarrow \boldsymbol{\xi}^{(3)} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

The general solution is

$$\mathbf{x}(t) = c_1 e^{-t} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + c_3 e^{2t} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

5. (15pts) For the given initial value problem

$$\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 2e^t \\ e^{3t} \end{pmatrix},$$

a fundamental matrix of the corresponding homogeneous system and its inverse are

$$\boldsymbol{\Psi} = \begin{pmatrix} e^{-3t} & e^{2t} \\ -4e^{-3t} & e^{2t} \end{pmatrix} \quad \text{and} \quad \boldsymbol{\Psi}^{-1} = \frac{1}{5} \begin{pmatrix} e^{3t} & -e^{3t} \\ 4e^{-2t} & e^{-2t} \end{pmatrix},$$

respectively. Find a particular solution and then the general solution.

Solution. A particular solution is calculated as follows

$$\int \boldsymbol{\Psi}^{-1} \mathbf{g}\, ds = \frac{1}{5} \int \begin{pmatrix} 2e^{4t} - e^{6t} \\ 8e^{-t} + e^t \end{pmatrix} ds = \frac{1}{5} \begin{pmatrix} \frac{1}{2}e^{4t} - \frac{1}{6}e^{6t} \\ -8e^{-t} + e^t \end{pmatrix} = \frac{1}{30} \begin{pmatrix} 3e^{4t} - e^{6t} \\ -48e^{-t} + 6e^t \end{pmatrix}.$$

$$\Longrightarrow \mathbf{x}_p = \frac{1}{30} \boldsymbol{\Psi} \begin{pmatrix} 3e^{4t} - e^{6t} \\ -48e^{-t} + 6e^t \end{pmatrix} = \frac{1}{6} \begin{pmatrix} -9e^t + e^{3t} \\ -12e^t + 2e^{3t} \end{pmatrix}.$$

The general solution is then

$$\mathbf{x} = \mathbf{x}_p + \mathbf{\Psi}\mathbf{c} = \frac{1}{6}\begin{pmatrix} -9e^t + e^{3t} \\ -12e^t + 2e^{3t} \end{pmatrix} + \begin{pmatrix} e^{-3t} & e^{2t} \\ -4e^{-3t} & e^{2t} \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

6. (25pts) Find the eigenvalues and eigenfunctions of

$$y'' + \lambda y = 0, \quad y(0) = 0, \quad y'(\pi) = 0,$$

- show that $\lambda = 0$ is not an eigenvalue and that the problem does not have negative eigenvalue,
- find all positive eigenvalues and the corresponding eigenfunctions.

solution. When $\lambda = 0$,

$$y'' = 0 \Longrightarrow y = c_1 + c_2 x,$$

which, together with the boundary condition, implies $y(x) = 0$ for all x. Hence, $\lambda = 0$ is not an eigenvalue.

Let $\lambda = -\sigma^2$ with $\sigma > 0$, the general solution is

$$y = c_1 e^{-\sigma x} + c_2 e^{\sigma x},$$

which, together with the boundary condition, implies

$$\begin{pmatrix} 1 & 1 \\ -e^{-\sigma\pi} & e^{\sigma\pi} \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = 0.$$

Since the coefficient matrix is nonsingular, $c_1 = c_2 = 0$. Hence, $y(x) = 0$ for all x. That is, the problem does not have negative eigenvalues.

Let $\lambda = \sigma^2$ with $\sigma > 0$, the general solution is computed as follows

$$\begin{cases} y = c_1\cos(\sigma x) + c_2\sin(\sigma x) \\ y' = \sigma\left(-c_1\sin(\sigma x) + c_2\cos(\sigma x)\right) \end{cases} \Longrightarrow \begin{cases} c_1 = 0 \\ c_2\cos(\sigma\pi) = 0 \end{cases} \Longrightarrow \sigma_k = \frac{2k+1}{2} \quad \text{for } k = 0, 1, 2, 3, \ldots$$

Hence, the eigenvalues are

$$\lambda_k = \left(\frac{2k+1}{2}\right)^2$$

and the corresponding eigenfunctions are

$$y_k(x) = \sin\left(\frac{2k+1}{2}x\right)$$

for $k = 0, 1, 2, 3, \ldots$

100	88	79	69	59	47
100	86	78	69	55	43
100	86	77	67	55	35
96	86	77	67	54	20
94	85	76	67	53	20
91	83	76	65	52	
90	83	75	63	52	
90	83	75	62	50	
	82	74	62	50	
	81	72	61		
	80	71			
	80				
	80				
8	13	11	10	9	5