Name: ____________________
PUID#: ____________________

Midterm 1A– Math 304 (2/18/10)
SHOW ALL RELEVANT WORK!!!

1. (20pts) Determine the radius of convergence of the given power series.

(1) $\sum\limits_{n=1}^{\infty} \frac{(3x+1)^n}{2^n}$: (1) 3, (2) 3/2, (3) 2, (4) 2/3, or (5) none of the above.

solution. (4)

$$L = \lim |\frac{(3x+1)^{n+1}}{2^{n+1}} / \frac{(3x+1)^n}{2^n}| = |3x+1|/2 < 1,$$

which implies

$$|3x+1| < 2 \Rightarrow |x + 1/3| < 2/3.$$

(2) $\sum\limits_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$: (1) 0, (2) 2, (3) 4, (4) ∞, or (5) none of the above.

solution. (4)

$$L = \lim |\frac{(-1)^{n+1}x^{2(n+1)}}{2^{2(n+1)}((n+1)!)^2} / \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}| = \lim |\frac{x^2}{2^2(n+1)^2}| = 0 < 1,$$

which implies it converges for all x. That is, the radius of convergence is ∞.

2. (15pts) For the differential equation $y'' - xy' - y = 0$ at $x_0 = 0$, (1) Find the recurrence relation, (2) find three terms in each of two linearly independent solutions (unless the series terminates sooner), and (3) if possible, find the general term in each solution.

solution. Let $y = \sum\limits_{n=0}^{\infty} a_n x^n$, then

$$\begin{aligned} 0 &= \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=0}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n \\ &= \sum_{n=0}^{\infty} \left((n+2)(n+1)a_{n+2} - (n+1)a_n\right) x^n. \end{aligned}$$

The recurrence relation is

$$a_{n+2} = \frac{a_n}{n+2} \quad \text{for} \quad n = 0, 1, 2, \cdots$$

which implies

$$a_2 = \frac{a_0}{2}, \quad a_4 = \frac{a_2}{4} = \frac{a_0}{4\cdot 2}, \quad a_6 = \frac{a_4}{6} = \frac{a_0}{6\cdot 4\cdot 2}, \cdots, \quad a_{2k} = \frac{a_0}{2\cdot 4 \cdots (2k)} = \frac{a_0}{2^k \cdot k!},$$

$$a_3 = \frac{a_1}{3}, \quad a_5 = \frac{a_3}{5} = \frac{a_1}{5\cdot 3}, \quad a_7 = \frac{a_5}{7} = \frac{a_1}{7\cdot 5\cdot 3}, \cdots, \quad a_{2k+1} = \frac{a_1}{3\cdot 5 \cdots (2k+1)} = \frac{a_1 2^k \cdot k!}{(2k+1)!}.$$

Now, two linearly independent solutions are

$$y_1 = \sum_{k=0}^{\infty} a_{2k} x^{2k} = a_0 \sum_{k=0}^{\infty} \frac{1}{2^k \cdot k!} x^{2k} \quad \text{and} \quad y_2 = \sum_{k=0}^{\infty} a_{2k+1} x^{2k+1} = a_1 \sum_{k=0}^{\infty} \frac{2^k \cdot k!}{(2k+1)!} x^{2k+1}.$$

3. (15pts) Determine the lower bounds for the radii of convergence of series solutions at $x_0 = 0$ and $x_0 = 4$ for

$$(2x^2 + 3x - 2)y'' + xy' + 4y = 0.$$

Solution. Set $0 = 2x^2 + 3x - 2 = (2x-1)(x+2)$. Then $x = 1/2$ and -2. At $x_0 = 4$,

$$dist(1/2, 4) = 7/2, \quad dist(-2, 4) = 6 \quad \Rightarrow \rho \geq 7/2.$$

At $x_0 = 0$,

$$dist(1/2, 0) = 1/2, \quad dist(-2, 0) = 2 \quad \Rightarrow \rho \geq 1/2.$$

4. (15pts) Let $y(x)$ be the solution of the following Euler equation:

$$x^2 y'' - 3xy' + 4y = 0, \quad y(1) = 2, \quad y'(1) = 3.$$

The value $y(e)$ is

(1) e^2, (2) $3e^2$, (3) e^{-2}, (4) $3e^{-2}$, or (5) none of the above.

Solution. (1) The charateristic equation is

$$0 = r(r-1) - 3r + 4 = r^2 - 4r + 4 = (r-2)^2 \quad \Rightarrow \quad r = r_1 = r_2 = 2,$$

which gives the general solution

$$y(x) = c_1 x^2 + c_2 x^2 \ln x \quad \text{and} \quad y' = 2c_1 x + c_2(2x \ln x + x).$$

The initial conditions imply

$$2 = y(1) = c_1 \quad \text{and} \quad 3 = y'(1) = 4 + c_2 \quad \Rightarrow \quad c_1 = 2 \text{ and } c_2 = -1.$$

Hence,

$$y(x) = x^2(2 - \ln x) \quad \Rightarrow \quad y(e) = (e)^2(2 - \ln e) = e^2.$$

5. (15pts) Let S_r and S_i denote the sets of regular and irregular singular points of the given differential equations

$$x(1-x^2)^3y'' + (1-x^2)^2y' + 2(1+x)y = 0,$$

respectively. Then

(1) $S_r = \{-1,\,0\}$ and $S_i = \{1\}$

(2) $S_r = \{-1,\,1\}$ and $S_i = \{0\}$

(3) $S_r = \{0,\,1\}$ and $S_i = \{-1\}$

(4) $S_r = \{-1,\,0,\,1\}$ and $S_i = \emptyset$

(5) $S_r = \{0\}$ and $S_i = \{1\}$.

solution. (1) Let $x(1-x^2)^3 = 0$. Then $x = -1,\,0,\,1$ are singular points. Since at $x = -1$

$$\lim_{x\to-1}(x+1)\frac{(1-x^2)^2}{x(1-x^2)^3} = \lim_{x\to-1}\frac{1}{x(1-x)} = -\frac{1}{2},\ \lim_{x\to-1}(x+1)^2\frac{2(1+x)}{x(1-x^2)^3} = \lim_{x\to-1}\frac{2}{x(1-x)^3} = -\frac{1}{4},$$

at $x = 0$

$$\lim_{x\to0} x\frac{(1-x^2)^2}{x(1-x^2)^3} = \lim_{x\to0}\frac{1}{1-x^2} = 1 \quad\text{and}\quad \lim_{x\to0} x^2\frac{2(1+x)}{x(1-x^2)^3} = \lim_{x\to0}\frac{2x(1+x)}{(1-x^2)^3} = 0$$

and at $x = 1$

$$\lim_{x\to1}(x-1)\tfrac{(1-x^2)^2}{x(1-x^2)^3} = \lim_{x\to1}\tfrac{-1}{x(x+1)} = -2,$$

$$\lim_{x\to1}(x-1)^2\tfrac{2(1+x)}{x(1-x^2)^3} = \lim_{x\to1}\tfrac{2}{x(1+x)(1-x^2)} \text{ DNE}.$$

Hence, $x = -1,\,0$ are regular singular points and $x = 1$ is a irregular singular point. Hence,

$$S_r = \{-1,\,0\} \text{ and } S_i = \{1\}.$$

6. (20pts) For boundary value problem

$$y'' + \lambda y = 0, \quad y'(0) = 0, \quad y(\pi) = 0,$$

find all eigenvalues (assume that they are real) and eigenfunctions.

solution. When $\lambda = 0$,

$$y'' = 0 \Longrightarrow y = c_1 + c_2x,$$

which, together with the boundary condition, implies $y(x) = 0$ for all x. Hence, $\lambda = 0$ is not an eigenvalue.

Let $\lambda = -\sigma^2$ with $\sigma > 0$, the general solution is

$$y = c_1e^{-\sigma x} + c_2e^{\sigma x},$$

which, together with the boundary condition, implies

$$\begin{pmatrix} -1 & 1 \\ e^{-\sigma\pi} & e^{\sigma\pi} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = 0.$$

Since the coefficient matrix is nonsingular, $c_1 = c_2 = 0$. Hence, $y(x) = 0$ for all x. That is, the problem does not have negative eigenvalues.

Let $\lambda = \sigma^2$ with $\sigma > 0$, the general solution is computed as follows

$$\begin{cases} y = c_1 \cos(\sigma x) + c_2 \sin(\sigma x) \\ y' = \sigma\left(-c_1 \sin(\sigma x) + c_2 \cos(\sigma x)\right) \end{cases} \Longrightarrow \begin{cases} c_2 = 0 \\ c_1 \cos(\sigma\pi) = 0 \end{cases} \Longrightarrow \sigma_k = \frac{2k+1}{2} \quad \text{for } k = 0, 1, 2, 3, ...$$

Hence, the eigenvalues are

$$\lambda_k = \left(\frac{2k+1}{2}\right)^2$$

and the corresponding eigenfunctions are

$$y_k(x) = \cos\left(\frac{2k+1}{2} x\right)$$

for $k = 0, 1, 2, 3, ...$

Grade Distribution

97	97	97	95	95	94	90
84	83	82	82			
79	77	73	72	72		
69	67	67	62	62	60	60
59	59	50				
49	47	42				
28	27	27	24			