1. (20pts) Determine the radius of convergence of the given power series.

(1) \( \sum_{n=1}^{\infty} \frac{(2x+1)^n}{3^n} \): (1) 3, (2) 3/2, (3) 2, (4) 2/3, or (5) none of the above.

**solution.** (2)

\[
L = \lim_{n \to \infty} \left| \frac{(2x+1)^{n+1}}{3^{n+1}} / \frac{(2x+1)^n}{3^n} \right| = \left|2x+1\right|/3 < 1,
\]

which implies

\[ |2x+1| < 3 \Rightarrow \left| x + 1/2 \right| < 3/2. \]

(2) \( \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{2^{n+1}(n)!} \): (1) \( \infty \), (2) 2, (3) 4, (4) 0, or (5) none of the above.

**solution.** (1)

\[
L = \lim_{n \to \infty} \left| \frac{(-1)^{n+1} x^{2(n+1)}}{2^{2(n+1)}((n+1)!)^2} / \frac{(-1)^n x^{2n}}{2^{2n}(n)!^2} \right| = \lim_{n \to \infty} \left| \frac{x^2}{2^2(n+1)^2} \right| = 0 < 1,
\]

which implies it converges for all \( x \). That is, the radius of convergence is \( \infty \).

2. (15pts) For the differential equation \( y'' - xy' - y = 0 \) at \( x_0 = 0 \), (1) Find the recurrence relation, (2) find three terms in each of two linearly independent solutions (unless the series terminates sooner), and (3) if possible, find the general term in each solution.

**solution.** Let \( y = \sum_{n=0}^{\infty} a_n x^n \), then

\[
0 = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=0}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n
\]

\[
= \sum_{n=0}^{\infty} ((n+2)(n+1)a_{n+2} - (n+1)a_n) x^n.
\]

The recurrence relation is

\[
a_{n+2} = \frac{a_n}{n+2} \quad \text{for} \quad n = 0, 1, 2, \ldots
\]

which implies

\[
a_2 = \frac{a_0}{2}, \quad a_4 = \frac{a_2}{4} = \frac{a_0}{8}, \quad a_6 = \frac{a_4}{6} = \frac{a_0}{48}, \quad a_{2k} = \frac{a_0}{2^k(k!)},
\]

\[
a_3 = \frac{a_1}{3}, \quad a_5 = \frac{a_3}{5} = \frac{a_1}{15}, \quad a_7 = \frac{a_5}{7} = \frac{a_1}{105}, \quad a_{2k+1} = \frac{a_0}{3^k(k+1)!}, \quad a_{2k+1} = \frac{a_0}{3^k(k+1)!}. \]

Now, two linearly independent solutions are

\[ y_1 = \sum_{k=0}^{\infty} a_{2k} x^{2k} = a_0 \sum_{k=0}^{\infty} \frac{1}{k!} x^{2k} \quad \text{and} \quad y_2 = \sum_{k=0}^{\infty} a_{2k+1} x^{2k+1} = a_1 \sum_{k=0}^{\infty} \frac{2^k \cdot k!}{(2k+1)!} x^{2k+1}. \]

3. (15pts) Determine the lower bounds for the radii of convergence of series solutions at \( x_0 = 0 \) and \( x_0 = 4 \) for

\[(2x^2 + 3x - 2)y'' + xy' + 4y = 0.\]

Solution. Set \( 0 = 2x^2 + 3x - 2 = (2x - 1)(x + 2) \). Then \( x = 1/2 \) and \(-2\). At \( x_0 = 4 \),

\[ \text{dist}(1/2, 4) = 7/2, \quad \text{dist}(-2, 4) = 6 \quad \Rightarrow \quad \rho \geq 7/2. \]

At \( x_0 = 0 \),

\[ \text{dist}(1/2, 0) = 1/2, \quad \text{dist}(-2, 0) = 2 \quad \Rightarrow \quad \rho \geq 1/2. \]

4. (15pts) Let \( y(x) \) be the solution of the following Euler equation:

\[ x^2 y'' - 3xy' + 4y = 0, \quad y(1) = 2, \quad y'(1) = 3. \]

The value \( y(1/e) \) is

(1) \( e^2 \), (2) \( 3e^2 \), (3) \( e^{-2} \), (4) \( 3e^{-2} \), or (5) none of the above.

Solution. (4) The characteristie equation is

\[ 0 = r(r - 1) - 3r + 4 = r^2 - 4r + 4 = (r - 2)^2 \quad \Rightarrow \quad r = r_1 = r_2 = 2, \]

which gives the general solution

\[ y(x) = c_1 x^2 + c_2 x^2 \ln x \quad \text{and} \quad y' = 2c_1 x + c_2 (2x \ln x + x). \]

The initial conditions imply

\[ 2 = y(1) = c_1 \quad \text{and} \quad 3 = y'(1) = 4 + c_2 \quad \Rightarrow \quad c_1 = 2 \text{ and } c_2 = -1. \]

Hence,

\[ y(x) = x^2 (2 - \ln x) \quad \Rightarrow \quad y(1/e) = (1/e)^2 (2 - \ln(1/e)) = 3e^{-2}. \]

5. (15pts) Let \( S_r \) and \( S_i \) denote the sets of regular and irregular singular points of the given differential equations

\[ x^2 (1 - x)y'' + (x - 2)y' - 3xy = 0, \]

respectively. Then

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(1) $S_r = \{1, 2\}$ and $S_i = \{0\}$
(2) $S_r = \{1\}$ and $S_i = \{0\}$
(3) $S_r = \{0, 1\}$ and $S_i = \emptyset$
(4) $S_r = \{0, 1, 2\}$ and $S_i = \emptyset$
(5) $S_r = \{0\}$ and $S_i = \{1\}$.

**solution. (2)** Let $x^2(1 - x) = 0$. Then $x = 0, 1$ are singular points. Since at $x = 0$

$$\lim_{x\to 0} \frac{x - 2}{x^2(1 - x)} = \lim_{x\to 0} \frac{x - 2}{x(1 - x)} \text{ DNE and } \lim_{x\to 0} \frac{-3x}{x^2(1 - x)} = \lim_{x\to 0} \frac{-3x}{1 - x} = 0$$

and at $x = 1$

$$\lim_{x \to 1} \frac{x - 2}{x^2(1 - x)} = \lim_{x \to 1} \frac{2 - x}{x^2} = 1 \text{ and } \lim_{x \to 1} \frac{-3x}{x^2(1 - x)} = \lim_{x \to 1} \frac{-3(1 - x)}{x} = 0$$

$x = 0$ is a irregular singular point and $x = 1$ is a regular singular point. Hence,

$$S_r = \{1\} \text{ and } S_i = \{0\}.$$

6. For boundary value problem

$$y'' + \lambda y = 0, \quad y'(0) = 0, \quad y(\pi) = 0,$$

find all eigenvalues (assume that they are real) and eigenfunctions.

**solution.** When $\lambda = 0$,

$$y'' = 0 \implies y = c_1 + c_2 x,$$

which, together with the boundary condition, implies $y(x) = 0$ for all $x$. Hence, $\lambda = 0$ is not an eigenvalue.

Let $\lambda = -\sigma^2$ with $\sigma > 0$, the general solution is

$$y = c_1 e^{-\sigma x} + c_2 e^{\sigma x},$$

which, together with the boundary condition, implies

$$\begin{pmatrix} -1 & 1 \\ e^{-\sigma \pi} & e^{\sigma \pi} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = 0.$$

Since the coefficient matrix is nonsingular, $c_1 = c_2 = 0$. Hence, $y(x) = 0$ for all $x$. That is, the problem does not have negative eigenvalues.

Let $\lambda = \sigma^2$ with $\sigma > 0$, the general solution is computed as follows

$$\begin{cases} y = c_1 \cos(\sigma x) + c_2 \sin(\sigma x) \\ y' = \sigma (-c_1 \sin(\sigma x) + c_2 \cos(\sigma x)) \end{cases} \implies \begin{cases} c_2 = 0 \\ c_1 \cos(\sigma \pi) = 0 \implies \sigma_k = \frac{2k + 1}{2} \text{ for } k = 0, 1, 2, 3, \ldots \end{cases}$$
Hence, the eigenvalues are

\[ \lambda_k = \left( \frac{2k + 1}{2} \right)^2 \]

and the corresponding eigenfunctions are

\[ y_k(x) = \cos \left( \frac{2k + 1}{2} x \right) \]

for \( k = 0, 1, 2, 3, \ldots \)

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**Grade Distribution**

97 97 97 95 95 94 90 84 83 82 82 79 77 73 72 72 69 67 67 62 62 60 60 59 59 50 49 47 42 28 27 27 24