Name: ____________________
PUID#: ____________________

# Midterm 2– Math 304 (4/1/10)
## SHOW ALL RELEVANT WORK!!!

$$\cos(\frac{n\pi}{2}) = \begin{cases} 0, & n = 2k-1 \\ (-1)^k, & n = 2k \end{cases} \quad \text{and} \quad \sin(\frac{n\pi}{2}) = \begin{cases} (-1)^{k+1}, & n = 2k-1 \\ 0, & n = 2k \end{cases}$$

for $k = 1, 2, ...$

1. (16pts) Let $f(x) = L - x$ for $-L < x < L$ and $f(x + 2L) = f(x)$.

6 (a) find a formula for $f(x)$ in the interval $-3L < x < -2L$;

$-3L < x < -2L \Rightarrow -L < x+2L < 0$

$f(x) = f(x+2L) = L - (x+2L) = -x - L$

10 (b) find the Fourier series.

2 $a_0 = \frac{1}{L}\int_{-L}^{L} f(x)dx = \frac{1}{L}\int_{-L}^{L}(L-x)dx = \frac{1}{L}\left[\int_{-L}^{L} L\,dx - \int_{-L}^{L} x\,dx\right] = 2L$

3 $a_n = \frac{1}{L}\int_{-L}^{L}(L-x)\cos\frac{n\pi x}{L}dx = \frac{1}{L}\left[\int_{-L}^{L} L\cos\frac{n\pi x}{L}dx - \int_{-L}^{L} x\cos\frac{n\pi x}{L}dx\right] = \int_{-L}^{L}\cos\frac{n\pi x}{L}dx = 0$

3 $b_n = \frac{1}{L}\left[\int_{-L}^{L} L\sin\frac{n\pi x}{L}dx - \int_{-L}^{L} x\sin\frac{n\pi x}{L}dx\right] = -\frac{2}{L}\int_{0}^{L} x\sin\frac{n\pi x}{L}dx$

$$\boxed{\begin{matrix} u = x & v' = \sin\frac{n\pi x}{L} \\ u' = 1 & v = -\frac{L}{n\pi}\cos\frac{n\pi x}{L} \end{matrix}}$$

$\int u'v = -\frac{L}{n\pi}\cdot\frac{L}{n\pi}\sin\frac{n\pi x}{L}$

$= -\frac{2}{L}\left[-\frac{L}{n\pi}x\cos\frac{n\pi x}{L} + \left(\frac{L}{n\pi}\right)^2\sin\frac{n\pi x}{L}\right]_0^L$

$= \frac{2L}{n\pi}\cos n\pi = (-1)^n\frac{2L}{n\pi}$ for $n = 1, 2, \cdots$

2 $f(x) = L + \sum_{n=1}^{\infty}(-1)^n\frac{2L}{n\pi}\sin\frac{n\pi x}{L}$

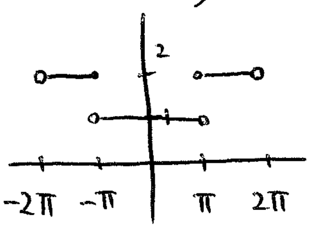
2. (18pts) Let $f(x) = \begin{cases} 1, & 0 < x < \pi, \\ 2, & \pi \le x < 2\pi. \end{cases}$

(a) Sketch the graphs of its even and odd extensions of $4\pi$.

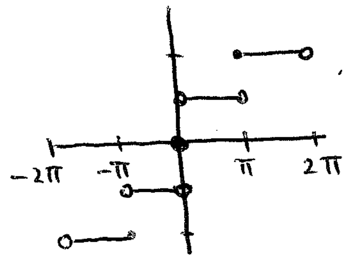
(b) Find its cosine series.

3 3

(a)




even extension




odd extension

(b)

3

$$a_0 = \frac{2}{2\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\left[\int_0^{\pi} 1\,dx + \int_{\pi}^{2\pi} 2\,dx\right] = 3$$

$$a_n = \frac{1}{\pi}\left[\int_0^{\pi} \cos\frac{nx}{2}\,dx + 2\int_{\pi}^{2\pi} \cos\frac{nx}{2}\,dx\right]$$

6

$$= \frac{1}{\pi}\left[\frac{2}{n}\sin\frac{nx}{2}\Big|_0^{\pi} + \frac{4}{n}\sin\frac{nx}{2}\Big|_{\pi}^{2\pi}\right]$$

$$= \frac{1}{n\pi}\left[2\sin\frac{n\pi}{2} + 4\left(0 - \sin\frac{n\pi}{2}\right)\right]$$

$$= -\frac{2}{n\pi}\sin\frac{n\pi}{2} = \begin{cases} (-1)^k \dfrac{2}{(2k-1)\pi}, & n = 2k-1 \\ 0 & n = 2k \end{cases}$$

3

$$f(x) = \frac{3}{2} + \frac{2}{\pi}\sum_{k=1}^{\infty} (-1)^k \frac{1}{2k-1}\cos\frac{(2k-1)x}{2}$$

3. (18pts) Determine whether the method of separation of variables can be used to replace the given partial differential equation by a pair of ordinary differential equations. If so, find the equations.

(a) $u_{xx} + u_{yy} + y^3 u = 0$ $\qquad u = X(x)Y(y)$

$$0 = X''Y + XY'' + y^3 XY = X''Y + X\left(Y'' + y^3 Y\right)$$

$$\Rightarrow \frac{X''}{X} + \frac{Y'' + y^3 Y}{Y} = 0 \Rightarrow \frac{X''}{X} = \sigma = -\left(\frac{Y'' + y^3 Y}{Y}\right)$$

yes

$$\begin{cases} X'' - \sigma X = 0 \\ Y'' + (y^3 + \sigma)Y = 0 \end{cases}$$

(b) $[p(x)u_x]_x - r(x)u_{tt} = 0$ $\qquad u = X(x)T(t)$

$$0 = (pX'T)_x - rXT'' = (pX')'T - rXT''$$

$$\Rightarrow \frac{(pX')'}{X} = \sigma = r\frac{T''}{T} \Rightarrow \begin{cases} (pX')' - \sigma X = 0 \\ rT'' - \sigma T = 0 \end{cases}$$

yes

(c) $u_{xx} + (x+t)u_t = 0$ $\qquad u = X(x)T(t)$

$$0 = X''T + (x+t)XT'$$
$$= X''T + xXT' + tXT'$$

no

4. (17pts) Consider the conduction of heat in a rod 10 cm in length. Suppose that $\alpha^2 = 4$. Find the temperature $u(x,t)$ at the middle of the rod if the initial temperature distribution in the rod is given by

$$u(x,0) = \begin{cases} 0, & 0 \le x < 5 \\ 10, & 5 \le x < 10 \end{cases}$$

and if the temperature at ends is given by

$$u(0,t) = 0 \quad \text{and} \quad u(10,t) = 10.$$

2

$L = 10,\ \alpha = 2,\ f(x) = u(x,0)\ ,\quad v(x) = \dfrac{T_2 - T_1}{L}x + T_1 = x$ 3

$$u(x,t) = x + \sum_{n=1}^{\infty} b_n e^{-\left(\frac{n\pi}{5}\right)^2 t} \sin\frac{n\pi x}{10}$$ 3

$$u(5,t) = 5 + \sum_{n=1}^{\infty} b_n e^{-\left(\frac{n\pi}{5}\right)^2 t} \sin\frac{n\pi}{2} = 5 + \sum_{k=1}^{\infty} b_{2k-1} e^{-\left(\frac{(2k-1)\pi}{5}\right)^2 t} \sin\frac{(2k-1)\pi}{2}$$

$$b_{2k-1} = \frac{2}{10}\int_0^{10} \left(f(x) - x\right) \sin\frac{(2k-1)\pi x}{10}\,dx \;\cancel{= \frac{1}{5}\left\{-\int_0^5 x\sin\frac{(2k-1)\pi x}{10}dx + 10\int_5^{10}\sin\frac{(2k-1)\pi x}{10}dx - \int_5^{10} x\sin\frac{(2k-1)\pi x}{10}dx\right\}}$$

$$= \frac{1}{5}\left\{10\int_5^{10}\sin\frac{(2k-1)\pi x}{10}dx - \int_0^{10} x\sin\frac{(2k-1)\pi x}{10}dx\right\}$$

$$= \frac{1}{5}\left\{-\frac{100}{(2k-1)\pi}\cos\frac{(2k-1)\pi x}{10}\Bigg|_5^{10} + \frac{10x}{(2k-1)\pi}\cos\frac{(2k-1)\pi x}{10}\Bigg|_0^{10} + \left(\frac{10}{(2k-1)\pi}\right)^2 \sin\frac{-(2k-1)\pi x}{10}\Bigg|_0^{10}\right\}$$

$u = x,\quad v' = \sin\frac{n\pi x}{10}$

$u' = 1,\quad v = -\frac{10}{n\pi}\cos\frac{n\pi x}{10}$

$\int u'v = -\left(\frac{10}{n\pi}\right)^2 \sin\frac{-n\pi x}{10}$

$$= \frac{1}{5}\left\{-\frac{100}{(2k-1)\pi}\cos(2k-1)\pi + \frac{100}{(2k-1)\pi}\left[\cos(2k-1)\pi\right]\right\}$$

3

$= 0 \qquad b_{2k} = ?$ 3

$\Rightarrow\ u(5,t) = 5$ 3

5. (17pts) Consider an elastic string of length 10 whose ends are held fixed. The string is set in motion from its equilibrium position with an initial velocity $u_t(x,0) = 1$. Let $a = 1$, find the displacement $u(x,t)$.

$L = 10, \quad g = u_t(x,0) = 1, \quad f = u(x,0) = 0,$

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{10} \sin\frac{n\pi t}{10}, \qquad \frac{n\pi}{10} c_n = \frac{2}{10}\int_0^{10} g \sin\frac{n\pi x}{10}\,dx$$

$$\Rightarrow c_n = \frac{2}{n\pi}\int_0^{10} \sin\frac{n\pi x}{10}\,dx = -\frac{20}{(n\pi)^2}\cos\frac{n\pi x}{10}\Big|_0^{10}$$

$$= -\frac{20}{(n\pi)^2}\left\{\cos n\pi - 1\right\} = -\frac{20}{(n\pi)^2}\left\{(-1)^n - 1\right\}$$

$$= \begin{cases} 0, & n = 2k \\ \dfrac{40}{(2k-1)^2\pi^2}, & n = 2k-1 \end{cases}$$

$$u(x,t) = \frac{40}{\pi^2}\sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} \sin\frac{(2k-1)\pi x}{10} \sin\frac{(2k-1)\pi t}{10}$$

6. (10pts) Using the method of separation of variables to find the solution of Laplace's equation, $u_{xx} + u_{yy} = 0$, in the rectangle $0 < x < a$, $0 < y < b$, also satisfying the boundary conditions

$$\begin{array}{lll} u(x,0) = 0, & u(x,b) = 0, & 0 < x < a, \\ u(0,y) = 0, & u(a,y) = f(y), & 0 \le y \le b. \end{array}$$

$u(x,y) = X(x)Y(y)$, $\quad X''Y + XY'' = 0 \Rightarrow -\frac{X''}{X} = \frac{Y''}{Y} = \sigma$

$X'' + \sigma X = 0$, $X(0) = 0$

$$\begin{cases} Y'' - \sigma Y = 0 \\ Y(0) = Y(b) = 0 \end{cases} \Longrightarrow \begin{array}{ll} \sigma_n = -\left(\frac{n\pi}{b}\right)^2 & \text{eigenvalues} \\ Y_n = \sin\frac{n\pi y}{b} & \text{eigenfunctions} \end{array} \quad \text{for } n = 1, 2, \cdots$$

$$\begin{cases} X_n'' + \sigma_n X_n = 0 \\ X_n(0) = 0 \end{cases} \Longrightarrow X_n = c_n e^{\frac{n\pi}{b}x} + d_n e^{-\frac{n\pi}{b}x} \quad \text{and } c_n + d_n = 0$$

$$= 2c_n \sinh\left(\frac{n\pi x}{b}\right)$$

$$u(x,y) = \sum_{n=1}^{\infty} c_n \sinh\left(\frac{n\pi x}{b}\right) \sin\frac{n\pi y}{b}$$

$$f(y) = \sum_{n=1}^{\infty} c_n \sinh\left(\frac{n\pi a}{b}\right) \sin\frac{n\pi y}{b}$$

$$\Rightarrow c_n = \frac{2}{b} \sinh^{-1}\left(\frac{n\pi a}{b}\right) \int_0^b f(y) \sin\frac{n\pi y}{b}\, dy.$$

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|---|---|---|---|---|---|---|---|---|
| 93 | 86 | 78 | 69 | 58 | 49 | 39 | 29 | 17 |
| 91 | 85 | 76 | 65 | 56 | 49 | 37 | | |
| | 82 | 76 | 63 | 50 | | 32 | | |
| | 80 | 74 | 63 | | | | | |
| | | 73 | 63 | | | | | |
| | | 73 | 61 | | | | | |
| | | 72 | | | | | | |
| | | 71 | | | | | | |
| | | 70 | | | | | | |
| 2 | 4 | 10 | 6 | 3 | 2 | 3 | 1 | 1 |