

MATLAB.1

Texts:

[B-D] Boyce & DiPrima, Elementary Diff. Eq. & B.V. Problems, 7th Ed
 [P-A] Polking & Arnold, ODEs using MATLAB, 2nd Ed

A. Getting Started with MATLAB

Read Chapters 1 and 3 in Polking. Do this before trying to go on.

B. More on Functions

Once a function $y=g(x)$ is defined with a M-file you now know how to graph it on an interval $[a,b]$ using the plot command. First you partition $[a,b]$. Next you evaluate g at these values. The plot command then plots these vertices connecting them by lines to give a piecewise linear approximation to the graph of g .

example: Plot $\sin(x^2)$ on $[-1,4]$. First we make a M-file.

```
function y=g(x)
y=sin(x.^2);
```

Go to the command window and type:

```
x=-1:.1:4;
plot(x,g(x))
title('y=sin(x^2)')
```

Note $x=[-1,-.9,\dots,3.9,4]$ and $g(x)=[g(-1),g(-.9),\dots,g(3.9),g(4)]$.

The graph is pretty crude because g oscillates more as x increases. A second way to graph is with the command fplot.

example: (continued)

Go to the command window and type:

```
fplot('g(x)',[-1,4])
```

One of the conveniences of fplot is, given $g.m$, it determines how fine the partition of $[a,b]$ needs to be (what the vector x should be) so that the piecewise linear approximation gives an accurate picture. A second feature of fplot is that it does not require $g.m$ to be written with array smart notation as is required when using plot.

CAUTION!! Don't name a function $y(x)$ (that is, use the file $y.m$); fplot will not plot it. Also fplot only recognizes "x" as the independent variable. So type "fplot('g(x)',[-1,4])", not "fplot('g(t)',[-1,4])" even if you used "t" as the independent variable in $g.m$.

ASSIGNMENT 1 :

$$A. \ a) \ \text{Let } A = \begin{bmatrix} 1 & 7 \\ 0 & -3 \end{bmatrix} \quad B = \begin{bmatrix} -3 & 2 \\ 3 & -2 \end{bmatrix} \quad C = \begin{bmatrix} 3 & 4 & 1 \\ -2 & -1 & 4 \\ 0 & -4 & 3 \end{bmatrix}$$

$$v = [3 \ 5]' \quad w = [2 \ -9]' \quad x = [-4 \ 3] \quad y = [0 \ 6 \ -3].$$

Note $[\]'$ is the transpose of $[\]$.

Try to find the following combinations in MATLAB. Which are defined and which are not? Where a combination is not defined explain why.

$$A*A \quad A*B \quad A*C \quad A.*A \quad A+B \quad A+C \quad A./C \quad A.\backslash B$$

$$A*x \quad x*A \quad v*A \quad x.*A \quad A.*x \quad y*C \quad C*x \quad C*y$$

$$A*v \quad A*w \quad x+v \quad v+w \quad x*x \quad v.*w \quad w.*v \quad y.^2$$

$$y.^2 \quad A.^2 \quad x+y \quad v*y \quad v.*y \quad v*w \quad x*w \quad x.*w$$

b) Show that $A = \begin{bmatrix} 1 & -2 \\ 0 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & -3 \\ 0 & 1 \end{bmatrix}$ commute.

c) Consider the initial value problem

$$\begin{aligned} z'(x) + 4z(x) &= 0 \\ z(0) &= 1 \\ z(0) &= -2. \end{aligned}$$

From Section 5.2 in [B-D] we have a power series representation for $z(x)$

$$z = a_0 + a_1x + a_2x^2 + \dots$$

where $a_0 = z(0) = 1$ and $a_1 = z'(0) = -2$.

From Math 262 you can solve the i.v.p. exactly,

$$z = \cos(2x) - \sin(2x) \quad (\text{for a review of this see Chapter 3 in [B-D]}).$$

First find a_2 and a_3 . Then make M.files for the three functions

$$z_2(x) = a_0 + a_1x + a_2x.^2, \quad z_3(x) = a_0 + a_1x + a_2x.^2 + a_3x.^3,$$

and $z(x) = \cos(2x) - \sin(2x)$ for x in $[0, \pi/2]$.

Plot the three curves on the same figure. Use different line styles for each curve, and label the figure appropriately.

B. Graph $f(x) = \cos(x.^4)$ on $[0, 2]$.

- Use the plot command with subintervals of length $h = .2$.
- Use fplot.

FOR ALL OF YOUR MATLAB GRAPHING HOMEWORK FOR THIS COURSE TITLE YOUR GRAPHS .

Minimization

To find where a function attains its minimum on an interval we use the fmin command.

example

First we make a M-file.

```
*****  
function w= f(x)  
w=exp(-x^2+2*x) -.5;  
*****
```

To find where f attains its minimum on [-1,3] go to the command window and type:

```
*****  
xm=fmin('f(x)',-1,3)  
*****
```

To find the minimum value type:

```
*****  
f(xm)  
*****
```

NOTE f(x) maximizes where -f(x) minimizes. To find the maximum value of f(x) on [-1,3] type:

```
*****  
xM=fmin('-f(x)',-1,3);  
f(xM)  
*****
```

To find the maximum of abs(f(x)) on [-1,3] type:

```
*****  
xM=fmin('-abs(f(x))',-1,3);  
abs(f(xM))  
*****
```

ASSIGNMENT 2 :

Let

$$h(x)=x^3-6*x^2+x .$$

Graph h(x) on [-1,5] .

Find where h(x) maximizes on [-1,5] and its maximum value.

Find where h(x) minimizes on [-1,5] and its minimum value.

Periodic Functions

We say $f(x)$ is periodic with period m if $f(x)=f(x+m)$ for all x .
Usually we are given a formula for f on a single period $[a,b)$ with $b-a=m$.
To find $f(x)$ for x not in $[a,b)$ we first find the greatest integer multiple
of m less than or equal to $x-a$,

$$z=m*\text{floor}((x-a)/m)$$

(Here $\text{floor}(t)$ is the MATLAB function giving the greatest integer $\leq t$.)
Now $y=x-z$ is in $[a,b)$. The point is that $f(y)=f(x)$ and we have a formula
for $f(y)$.

example

Let $f(x)=\text{abs}(x)$ for $-3\leq x<3$ and assume $f(x)$ is $3-(-3)=6$ -periodic.
To write a M-file for f type :

```
*****  
function w=f(x)  
z=6*floor((x+3)/6);  
y=x-z;  
w=abs(y);  
*****
```

ASSIGNMENT 3 :

1. Let
 $f(x)=x^2$ if $-2\leq x<2$
and $f(x)$ is 4-periodic.
Write $f.m$ and graph $f(x)$ on $[-10,10]$.
2. Let
 $g(x)=x$ if $-2\leq x<2$
and $g(x)$ is 4-periodic .
Write $g.m$ and graph $g(x)$ on $[-10,10]$.

Linear Systems

Consider

$$X' = AX$$

where A is nxn .If there are n linearly independent eigenvectors for A , $\{v(1), \dots, v(n)\}$, then a fundamental set of solutions is given by $\{\exp(s(i)*t)v(i), i=1..n\}$. If n is large though this may be hard to find by hand. Here we show how to use Matlab to find the eigenvalues and eigenvectors.

example

Let

A =

$$\begin{bmatrix} -1 & 0 & 2 \\ 2 & 3 & 6 \\ -2 & 0 & -1 \end{bmatrix}$$

Type :

[E,V]=eig(A)

We get

E =

$$\begin{bmatrix} 0 & 0.5000 & 0.5000 \\ 1.0000 & 0.1000- 0.7000i & 0.1000+ 0.7000i \\ 0 & 0+ 0.5000i & 0- 0.5000i \end{bmatrix}$$

V =

$$\begin{bmatrix} 3.0000 & 0 & 0 \\ 0 & -1.0000+ 2.0000i & 0 \\ 0 & 0 & -1.0000- 2.0000i \end{bmatrix}$$

The diagonal matrix V lists the eigenvalues of A and the corresponding columns of E are the eigenvectors. Using this information we can write down 3 linearly independent real solutions (a fundamental set).

$$X(1) = \exp(3*t) * [0, 1, 0]'$$

$$X(2) = \exp(-t) * [0.5*\cos(2*t), 0.1*\cos(2*t) + \sin(2*t)*0.7, -0.5*\sin(2*t)]'$$

$$X(3) = \exp(-t) * [0.5*\sin(2*t), -0.7*\cos(2*t) + 0.1*\sin(2*t), 0.5*\cos(2*t)]'$$

Note v' is the transpose of v .

What do you do if there are not n linearly independent eigenvectors? To see how to complete a fundamental set in this case look at Polking, pages 163-170..

ASSIGNMENT 4 :

Write down a real fundamental set if

1.

A =

-0.2	0	0.2
0.2	-0.4	0
0	0.4	-0.2

2.

A =

4	1	1	7
1	4	10	1
1	10	4	1
7	1	1	4

MATLAB.5

Read Chapter 7 in Polking.

ASSIGNMENT 5 :

1. Use ode45 to solve the initial value problem

$$x'' - x'/t + (4t^2)x = 0 \quad \text{where } x=x(t) \quad , \quad 1 < t < 3$$

and

$$x(1)=1, \quad x'(1)=3. \quad (\text{Look at page 89.})$$

Plot $x(t)$ on the interval.

2. Predator-Prey Problem

a) During the 1st World War fisherman noticed a larger percentage of their catch was sharks

and other fish eating fish than was true before or after the war.

Let $x=x(t)$ denote the population of prey and $y=y(t)$ the population of predators. From section

9.6 in B&D we can use the model

$$(1) \quad \begin{aligned} x' &= a*x - b*x*y, \\ y' &= -c*y + d*x*y \end{aligned}$$

for some positive constants. Solve this with $a=0.4, b=0.01, c=0.3, d=0.005$ and initial conditions $x(0)=50, y(0)=30$ for $0 < t < 100$.

Plot x versus t, y versus t , and a x versus y phase plane plot. You can use either ode45 or pplane for this.

b) The system (1) does not take fishing into account. If you fish you catch both x and y and the model becomes

$$(2) \quad \begin{aligned} x' &= a*x - b*x*y - e*x, \\ y' &= -c*y + d*x*y - e*y \end{aligned}$$

for some constant e . Solve (2) with the same a, b, c, d and initial conditions as before for

$e = 0, 0.01, 0.02, 0.03, \text{ and } 0.04$. Plot the 5 solutions on the same phase plane ($x-y$)

and label them properly. You will have to use ode45 for this (solve the 5 initial value

problems and plot them on the same graph).

c) Can you use the plot from b) to explain why more sharks were caught during WWI ?

You can assume there was less fishing during the war.

Plotting Fourier Series

We plot the partial sums for the Fourier series for $\text{abs}(x)$ on $[-3,3]$.

First we compute the coefficients by hand.

$$\begin{aligned} a(0) &= 3, \\ a(m) &= -6 \cdot (1 - (-1)^m) / (m \cdot \pi)^2 \quad \text{for } m \geq 1, \\ b(m) &= 0. \end{aligned}$$

Next we make an M-file for the n th partial sum.

```
function w=summ(x,n)

w=3/2;
for m=1:n
    w=w-cos(m*pi*x/3)*6*(1-(-1)^m)/(m*pi)^2;
end
```

We want to compare $\text{summ}(x,n)$ with the 6-periodic extension of $\text{abs}(x)$ (See the example in HW.3) . Its M-file is

```
function w=f(x)
z=6*floor((x+3)/6);
y=x-z;
w=abs(y);
*****
```

Now we go to the command window and plot $\text{summ}(x,12)$ and $f(x)$ on $[-6,7]$. We use "hold on" and "hold off" to plot the graphs together.

```
hold on
fplot('summ(x,12)', [-6,7])
fplot('f(x)', [-6,7])
hold off;
```

To see how far apart the graphs are find the maximum of $\text{abs}(f(x) - \text{summ}(x,12))$. (See HW.2) .

ASSIGNMENT 6 :

Let $f(x) = x^2$ for $-2 \leq x < 2$.

Find the Fourier series for f on $[-2,2]$. Plot the partial series for $f, \text{summ}(x,8)$ together with a plot of the 4-periodic extension of $f(x)$ on $[-5,6]$.

For what n is the maximum of $\text{abs}(f(x) - \text{summ}(x,n))$ on $[-5,6]$ less than 0.1?

Plotting Fourier Series

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end
```

We want to compare $\text{summ}(x,n)$ with the 6-periodic extension of $\text{abs}(x)$ (See the example in HW.3) . Its M-file is

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```
hold on
fplot('summ(x,12)', [-6,7])
fplot('f(x)', [-6,7])
hold off;
```

To see how far apart the graphs are find the maximum of $\text{abs}(f(x) - \text{summ}(x,12))$. (See HW.2) .

ASSIGNMENT 6 :

Let $f(x) = x^2$ for $-2 \leq x < 2$.

Find the Fourier series for f on $[-2,2]$. Plot the partial series for $f, \text{summ}(x,8)$ together with a plot of the 4-periodic extension of $f(x)$ on $[-5,6]$.

For what n is the maximum of $\text{abs}(f(x) - \text{summ}(x,n))$ on $[-5,6]$ less than 0.1?

Partial Differential Equations
Solving the Heat Equation .

Let $f(x)=0$ if $0<x<1$,
 $=1$ if $1<x<2$.

Consider the problem

$$\begin{aligned} d^2u/dx^2 &= du/dt & 0 < x < 2, \quad t > 0 \\ u(0, t) &= u(2, t) = 0 & t > 0, \\ u(x, 0) &= f(x) . \end{aligned}$$

The Fourier sine coefficients for f are
 $b(m) = (\cos(m\pi/2) - \cos(m\pi)) * 2/m/\pi$

The M-file computing the first n terms is

```
*****
function w=U(x,t,n)
w=0;
for m=1:n
    w=w+2*(cos(m*pi/2)-cos(m*pi))/m/pi*exp(-m^2*pi^2*t/4).*sin(m*pi*x/2);
end
*****
```

Notice the .* above. We will do a 3-dimensional plot and U.m needs to be array smart for this.

Let's set $n=50$.

To plot the temperature distribution at different times enter

```
*****
hold on
fplot('U(x,0,50)', [0,2])
fplot('U(x,.1,50)', [0,2])
fplot('U(x,.5,50)', [0,2])
fplot('U(x,1,50)', [0,2])
fplot('U(x,2,50)', [0,2])
fplot('U(x,5,50)', [0,2])
hold off
*****
```

To do a 3-D plot on the x - t rectangle $0 < x < 2, 0 < t < 1$ you subdivide it into rectangles of side length dx by dt and plot the u at the vertices.

Here we choose $dx=.05$ and $dt=.05$.

Enter

```
*****
x=0:.05:2;
t=0:.05:1;
[x,t]=meshgrid(x,t);
mesh(x,t,U(x,t,50))
*****
```

ASSIGNMENT 7 :

1. HEAT EQUATION

Let u model the temperature in a rod and solve

$$d^2u/dx^2 = du/dt \quad 0 < x < 40, t > 0$$

$$u(0,t) = u(40,t) = 0 \quad t > 0,$$

$$u(x,0) = x \quad \text{for } 0 < x < 40 .$$

- Find the series solution and consider the partial sum $U = U(x,t,50)$.
- Plot U versus x for $t = 5, 10, 20, 40, 100$ and 200 .
- For each value of t used in b) estimate the value of x for which the value of U is greatest . (See HW.2 .) Plot these values versus t to see how the warmest point in the rod changes with time.
- Plot U versus t for $x = 10, 20$ and 30 . The command `fplot` only recognizes "x" as an independent variable. For example if you have an M-file for a function $g(x,t)$, and you want the graph of $g(3,t)$ for $1 < t < 3$ type :
`fplot('g(3,x)', [1,3])` .
- Do a 3-D plot of U versus x and t .
- How long does it take for the entire rod to cool off to a temperature of no more than 2 degrees ?

2. WAVE EQUATION

Let u model the displacement of a vibrating string and solve

$$d^2u/dx^2 = d^2u/dt^2 \quad 0 < x < 10, t > 0$$

$$u(0,t) = u(10,t) = 0 \quad t > 0,$$

$$u(x,0) = \begin{cases} 1 & \text{for } 4 < x < 6 \\ 0 & \text{for } 0 < x < 4 \text{ \& } 6 < x < 10 \end{cases}$$

$$du/dt(x,0) = 0 \quad \text{for } 0 < x < 10 .$$

- Find the series solution and consider the partial sum $U = U(x,t,50)$.
- Plot U versus x for $t = 0, 2, 4, \dots, 18, 20$.Each on its own axes.
- Plot U versus t for $0 < t < 20$ for $x = 5$ and $x = 2.5$ on separate axes.
- Describe the motion of the string in a few sentences.

MATLAB.8

Numerical Methods First consider the ode

$$y' = f(x, y) = 2xy^2$$

$$y(0) = 0.1.$$

We analyze this using the Euler, Improved Euler and Runge-Kutta methods. First we make a M-file for the right hand side.

```
*****
function w=f(x,y)
w=2*x*y^2;
*****
```

Next we make M-files for each of the three methods.

```
*****
function [X,Y]=eu(x,y,xf,n)
h=(xf-x)/n;
X=x;Y=y;
for i=1:n
    y=y+h*f(x,y);
    x=x+h;
    X=[X;x];
    Y=[Y;y];
end
```

```
*****
function [X,Y]=imeul(x,y,xf,n)
h=(xf-x)/n;
X=x;Y=y;
for i=1:n
    k1=f(x,y);
    k2=f(x+h,y+h*k1);
    y=y+h*(k1+k2)/2;
    x=x+h;
    X=[X;x];
    Y=[Y;y];
end
```

```
*****
function [X,Y]=rk(x,y,xf,n)
h=(xf-x)/n;
X=x;Y=y;
for i=1:n
    k1=f(x,y);
    k2=f(x+h/2,y+h*k1/2);
    k3=f(x+h/2,y+h*k2/2);
    k4=f(x+h,y+h*k3);
    y=y+h*(k1+2*k2+2*k3+k4)/6;
    x=x+h;
    X=[X;x];
    Y=[Y;y];
end
```

Here (x,y) are the initial values, xf is the final x-value and n is the number of partitions.

[X,Y] is the (n+1) x 2 matrix representing the computed nodes. To plot all three you go to the command window and enter

```
*****
```

```
[z,w]=eu(0,0.1,3,20);
[s,t]=imeul(0,0.1,3,20);
[u,v]=rk(0,0.1,3,20);
plot(z,w,s,t,'--',u,v,'o')
```

Euler is given by a solid curve, Improved Euler by '--'s and Runge-Kutta by 'o's.

ASSIGNMENT 8 :

1. Let $f(x,y)$ be as above. Type in the M-files for f.m and the three approximations.

Plot their graphs for $n=20$ and $xf=3$.

The actual solution is $y(x)=1/(10-x^2)$.

To find the distance between $y(x)$ and the Euler approximation on the interval $[0,3]$ for a given value of n type :

```
x=0:3/n:3;
y=1./(10-x.^2);
y=y';
```

% Here we have transposed y from a row vector to a column vector.

```
[z,w]=eu(0,0.1,3,n);
max(abs(y-w))
```

Here

$\text{abs}(y-w)=[\text{abs}(y(1)-w(1)), \dots, \text{abs}(y(n+1)-w(n+1))]'$
and $\max(\text{abs}(y-w))$ is the maximum of the $n+1$ components.

The theory predicts that

$$(*) \quad \max(\text{abs}(y-w)) < C * (3/n)$$

for some constant C and each $n > 1$.

2. Set $C1(n) = \max(\text{abs}(y-w)) / (3/n)$.

Compute $C1(n)$ for $n=100, n=200, \dots, n=800$.

Does $C1(n)$ grow as n gets large or tend to level off? Is this consistent with (*) ?

Set $C2(n) = \max(\text{abs}(y--t)) / (3/n)^2$ for the Improved Euler method.

Compute $C2(n)$ for the same values of n .

What should $C3(n)$ be for the Runge-Kutta method? Compute $C3(n)$.