Name: __________________
PUID#: __________________

Midterm 1A– Math 362 (09/29/09)
SHOW ALL RELEVANT WORK!!!

1. (10pts) Suppose that a particle following the path, $\mathbf{c}(t) = (\sin e^t,\, t,\, 4 - t^3)$, flies off on a tangent at $t = 1$. The position of the particle at the time $t = 2$ is

(1) $(\sin e + 2e\cos e,\, 3,\, -3)$, (2) $(\sin e + e\cos e,\, 2,\, 0)$, (3) $(\sin 1 + 2\cos 1,\, 2,\, 4)$, (4) $(e\cos e + \sin e,\, 2,\, 3)$, and (5) None of the above.

solution. **(2)** The tangent line to the path at $t = 1$ is

$$\mathbf{l}(t) = \mathbf{c}(1) + (t-1)\mathbf{c}'(1),$$

where $\mathbf{c}'(t) = (e^t\cos e^t,\, 1,\, -3t^2)$. The position of the particle at the time $t = 2$ is

$$\mathbf{l}(2) = \mathbf{c}(1) + (2-1)\mathbf{c}'(1) = (\sin e,\, 1,\, 3) + (e\cos e,\, 1,\, -3) = (\sin e + e\cos e,\, 2,\, 0).$$

2. (10pts) Sketch the curve that is the image of the path

$$\mathbf{c}(t) = (|t|,\, |t + \frac{1}{3}|) \quad \text{where} \quad -1 \le t \le 1.$$

solution. By opening the absolute value, we have

$$\mathbf{c}(t) = \begin{cases} (-t,\, -t - 1/3) & -1 \le t \le -1/3, \\ (-t,\, t + 1/3) & -1/3 \le t \le 0, \\ (t,\, t + 1/3) & 0 \le t \le 1. \end{cases} = \begin{cases} (0,\, -1/3) + t(-1,\, -1) & -1 \le t \le -1/3, \\ (0,\, 1/3) + t(-1,\, 1) & -1/3 \le t \le 0, \\ (0,\, 1/3) + t(1,\, 1) & 0 \le t \le 1. \end{cases}$$

Hence, the curve is a three line segments.

3. (20pts) Given $g(x,y)=(x^2,\, y^2+1)$ and $f(u,v)=(u,\, u+v,\, v^2)$,
(a) write a formula for $f\circ g$;
(b) calculate $\mathbf{D}(f\circ g)(1,1)$ by the chain rule.

solution. (a) Since $u=x^2$ and $v=y^2+1$,

$$f\circ g(x,y)=(x^2,\, x^2+y^2+1,\, (y^2+1)^2).$$

solution. (b)

$$Df=\begin{pmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} \\ \frac{\partial f_3}{\partial u} & \frac{\partial f_3}{\partial v} \end{pmatrix}=\begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 2v \end{pmatrix} \quad \text{and} \quad Dg=\begin{pmatrix} \frac{\partial g_1}{\partial x} & \frac{\partial g_1}{\partial y} \\ \frac{\partial g_2}{\partial x} & \frac{\partial g_2}{\partial y} \end{pmatrix}=\begin{pmatrix} 2x & 0 \\ 0 & 2y \end{pmatrix}$$

Since $g(1,1)=(1,\,2)$, then

$$Df(g(1,1))=\begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 4 \end{pmatrix} \quad \text{and} \quad Dg(1,1)=\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

which implies

$$\mathbf{D}(f\circ g)(1,1)=\begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 4 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}=\begin{pmatrix} 2 & 0 \\ 2 & 2 \\ 0 & 8 \end{pmatrix}.$$

Grade Distribution

100	1
$90's$	1
$80's$	7
$70's$	4
$60's$	11
$50's$	7
$40's$	4
$30's$	4
$20's$	2
total	41

5. (10pts) The directional derivative of $f(x, y, z) = xy^2 + y^2z^3 + z^3x$ along the direction $\mathbf{v} = \mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$ at $(1, 0, -1)$ is

(1) 5, (2) $5/\sqrt{14}$, (3) $(-1,\ 0,\ 6)$, (4) $(-1,\ 0,\ 6)/\sqrt{14}$, (5) None of the above.

solution. (2) The gradient is

$$\nabla f = (y^2 + z^3,\ 2xy + 2yz^3,\ 3y^2z^2 + 3z^2x),$$

and its value at $(1, 0, -1)$ is $\nabla f(1, 0, -1) = (-1,\ 0,\ 3)$. The unit vector along the direction $\mathbf{v} = \mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$ is $\frac{1}{\sqrt{14}}(1,\ 3,\ 2)$. Hence, the directional derivative is

$$(-1,\ 0,\ 3) \cdot \frac{1}{\sqrt{14}}(1,\ 3,\ 2) = \frac{5}{\sqrt{14}}.$$

6. (10pts) Find the tangent plane to the hyperboloid $x^2 + y^2 - z^2 = 18$ at $(3,\ 5,\ -4)$.

solution. Let $f(x, y, z) = x^2 + y^2 - z^2$, then its gradient at $(3,\ 5,\ -4)$ is

$$\nabla f(3,\ 5,\ -4) = 2(x, y, -z)|_{(3,\,5,\,-4)} = 2(3,\ 5,\ 4).$$

The tangent plane at $(3,\ 5,\ -4)$ is

$$0 = (3,\ 5,\ 4) \cdot (x - 3, y - 5, z + 4) = 3(x - 3) + 5(y - 5) + 4(z + 4).$$

Or

$$3x + 5y + 4z = 18.$$

7. (10pts) A particle moves in such a way that its acceleration is constantly equal to $-\mathbf{k}$. If the position when $t = 0$ is $(0, 0, 1)$ and the velocity at $t = 0$ is $\mathbf{i} + \mathbf{j}$, when and where does the particle fall below the plane $z = 0$?

solution. Since $\mathbf{v}(t) = \int \mathbf{a}(t)\, dt$ and $\mathbf{c}(t) = \int \mathbf{v}(t)\, dt$, we have

$$\mathbf{a}(t) = -\mathbf{k} = (0,\ 0,\ -1) \Rightarrow \mathbf{v}(t) = (c_1,\ c_2,\ -t + c_3)$$

$$\mathbf{v}(0) = \mathbf{i} + \mathbf{j} = (1,\ 1,\ 0) = (c_1,\ c_2,\ c_3) \Rightarrow \mathbf{v}(t) = (1,\ 1,\ -t)$$

$$\mathbf{v}(t) = (1,\ 1,\ -t) \Rightarrow \mathbf{c}(t) = (t + c_1,\ t + c_2,\ -t^2/2 + c_3)$$

$$\mathbf{c}(0) = (0,\ 0,\ 1) = (c_1,\ c_2,\ c_3) \Rightarrow \mathbf{c}(t) = (t,\ t,\ -t^2/2 + 1)$$

Let $z(t) = 0$, we have $-t^2/2 + 1 = 0$, i.e., $t = \sqrt{2}$. Hence, the particle falls below the plane at $(\sqrt{2},\ \sqrt{2},\ 0)$ when $t = \sqrt{2}$.

8. (10pts) Show that, at a local maximum or minimum of $\|\mathbf{r}(t)\|$, the vector $\mathbf{r}'(t)$ is perpendicular to $\mathbf{r}(t)$. That is, $\mathbf{r}'(t) \cdot \mathbf{r}(t) = 0$ for all t.

proof. Let $f(t) = \|\mathbf{r}(t)\| = \sqrt{x^2(t) + y^2(t) + z^2(t)}$, then

$$\begin{aligned} f'(t) &= \frac{1}{2}\left(x^2(t) + y^2(t) + z^2(t)\right)^{-1/2} 2\left(x(t)x'(t) + y(t)y'(t) + z(t)z'(t)\right) \\ &= \frac{1}{\|\mathbf{r}(t)\|}\mathbf{r}'(t) \cdot \mathbf{r}(t). \end{aligned}$$

At a local maximum or minimum of $f(t) = \|\mathbf{r}(t)\|$, we have

$$0 = f'(t) = \frac{1}{\|\mathbf{r}(t)\|}\mathbf{r}'(t) \cdot \mathbf{r}(t)$$

which implies

$$\mathbf{r}'(t) \cdot \mathbf{r}(t) = 0.$$

9. (10pts) The arc length of the path $\mathbf{c}(t) = (\cos 3t,\, \sin 3t,\, 2t^{3/2})$ for $0 \le t \le 1$ is
(1) $3\sqrt{2}$, (2) $3(2-\sqrt{2})/2$, (3) $2(2\sqrt{2}-1)$, (4) $3(2\sqrt{2}-1)$, (5) None of the above.
solution. (3)

$$\mathbf{c}'(t) = 3(-\sin 3t,\, \cos 3t,\, t^{1/2}), \quad \|\mathbf{c}'(t)\| = 3\sqrt{1+t}$$

$$L = \int_0^1 \|\mathbf{c}'(t)\| dt = 3\int_0^1 \sqrt{1+t}dt = 2(1+t)^{3/2}|_0^1 = 2(2\sqrt{2}-1).$$

10. (10pts) Let $\mathbf{c}(t)$ be a given path, $a \le t \le b$. Let $s = \alpha(t)$ be a new variable, where α is a strictly increasing C^1 function given on $[a, b]$. For each s in $[\alpha(a), \alpha(b)]$, there is a unique t with $\alpha(t) = s$. Define the function $\mathbf{d}(s) : [\alpha(a), \alpha(b)] \to R^3$ by $\mathbf{d}(s) = \mathbf{c}(t)$. Show that $\mathbf{c}(t)$ and $\mathbf{d}(s)$ have the same arc length.

proof. The arc lengths for pathes $\mathbf{c}(t)$ and $\mathbf{d}(s)$ are

$$L_1 = \int_a^b \|\mathbf{c}'(t)\|\, dt \quad \text{and} \quad L_2 = \int_{\alpha(a)}^{\alpha(b)} \|\mathbf{d}'(s)\|\, ds,$$

respectively. By the change of variable: $s = \alpha(t)$, one has

$$ds = \alpha'(t)dt, \quad \mathbf{d}'(s) = \frac{d}{dt}(\mathbf{d}(\alpha(t))) \cdot \frac{dt}{ds} = \frac{1}{\alpha'(t)}\frac{d}{dt}(\mathbf{c}(t)) = \frac{1}{\alpha'(t)}\mathbf{c}'(t).$$

Since α is a strictly increasing C^1 function given on $[a, b]$, we have $\alpha'(t) > 0$ which implies

$$\|\mathbf{d}'(s)\| = \|\mathbf{c}'(t)\|\frac{1}{|\alpha'(t)|} = \frac{1}{\alpha'(t)}\|\mathbf{c}'(t)\|.$$

Now, we have

$$L_2 = \int_{\alpha(a)}^{\alpha(b)} \|\mathbf{d}'(s)\|\, ds = \int_a^b \frac{1}{\alpha'(t)}\|\mathbf{c}'(t)\|\, \alpha'(t)dt = \int_a^b \|\mathbf{c}'(t)\|\, dt = L_1.$$

This completes the proof.