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**Midterm 1B– Math 362 (09/29/09)**  
**SHOW ALL RELEVANT WORK!!!**

1. (10pts) Suppose that a particle following the path,  $\mathbf{c}(t) = (\cos^2 t, 3t - t^3, 1/t)$ , flies off on a tangent at  $t = 1$ . The position of the particle at the time  $t = 2$  is

- (1)  $(\cos^2 1 - 2 \sin 2, 2, -1)$ , (2)  $(2 \cos^2 1 - \sin 2, 4, 1)$ , (3)  $(\cos^2 1 - \cos 2, 2, 0)$ ,  
(4)  $(\cos^2 1 - \sin 2, 2, 0)$ , and (5) None of the above.

**solution.** (4) The tangent line to the path at  $t = 1$  is

$$\mathbf{l}(t) = \mathbf{c}(1) + (t - 1)\mathbf{c}'(1),$$

where  $\mathbf{c}'(t) = (-2 \cos t \sin t, 3 - 3t^2, -t^{-2})$ . The position of the particle at the time  $t = 2$  is

$$\mathbf{l}(2) = \mathbf{c}(1) + (2 - 1)\mathbf{c}'(1) = (\cos^2 1, 2, 1) + (-\sin 2, 0, -1) = (\cos^2 1 - \sin 2, 2, 0).$$

2. (10pts) Sketch the curve that is the image of the path

$$\mathbf{c}(t) = (|t|, |t + \frac{1}{3}|) \quad \text{where} \quad -1 \leq t \leq 1.$$

**solution.** By opening the absolute value, we have

$$\mathbf{c}(t) = \begin{cases} (-t, -t - 1/3) & -1 \leq t \leq -1/3, \\ (-t, t + 1/3) & -1/3 \leq t \leq 0, \\ (t, t + 1/3) & 0 \leq t \leq 1. \end{cases} = \begin{cases} (0, -1/3) + t(-1, -1) & -1 \leq t \leq -1/3, \\ (0, 1/3) + t(-1, 1) & -1/3 \leq t \leq 0, \\ (0, 1/3) + t(1, 1) & 0 \leq t \leq 1. \end{cases}$$

Hence, the curve is a three line segments.

3. (20pts) Given  $g(x, y) = (x^2, y^2 + 1)$  and  $f(u, v) = (u, u + v, v^2)$ ,

(a) write a formula for  $f \circ g$ ;

(b) calculate  $\mathbf{D}(f \circ g)(1, 1)$  by the chain rule.

**solution. (a)** Since  $u = x^2$  and  $v = y^2 + 1$ ,

$$f \circ g(x, y) = (x^2, x^2 + y^2 + 1, (y^2 + 1)^2).$$

**solution. (b)**

$$Df = \begin{pmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} \\ \frac{\partial f_3}{\partial u} & \frac{\partial f_3}{\partial v} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 2v \end{pmatrix} \quad \text{and} \quad Dg = \begin{pmatrix} \frac{\partial g_1}{\partial x} & \frac{\partial g_1}{\partial y} \\ \frac{\partial g_2}{\partial x} & \frac{\partial g_2}{\partial y} \end{pmatrix} = \begin{pmatrix} 2x & 0 \\ 0 & 2y \end{pmatrix}$$

Since  $g(1, 1) = (1, 2)$ , then

$$Df(g(1, 1)) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 4 \end{pmatrix} \quad \text{and} \quad Dg(1, 1) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

which implies

$$\mathbf{D}(f \circ g)(1, 1) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 2 & 2 \\ 0 & 8 \end{pmatrix}.$$

### Grade Distribution

100	1
90's	1
80's	7
70's	4
60's	11
50's	7
40's	4
30's	4
20's	2
total	41

5. (10pts) The directional derivative of  $f(x, y, z) = xy^2 + y^2z^3 + z^3x$  along the direction  $\mathbf{v} = \mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$  at  $(1, -1, 0)$  is

(1)  $(1, -6, 0)/\sqrt{14}$ , (2)  $(1, -6, 0)$ , (3)  $-5$ , (4)  $-5/\sqrt{14}$ , (5) None of the above.

**solution.** (4) The gradient is

$$\nabla f = (y^2 + z^3, 2xy + 2yz^3, 3y^2z^2 + 3z^2x),$$

and its value at  $(1, -1, 0)$  is  $\nabla f(1, -1, 0) = (1, -2, 0)$ . The unit vector along the direction  $\mathbf{v} = \mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$  is  $\frac{1}{\sqrt{14}}(1, 3, 2)$ . Hence, the directional derivative is

$$(1, -2, 0) \cdot \frac{1}{\sqrt{14}}(1, 3, 2) = \frac{-5}{\sqrt{14}}.$$

6. (10pts) Find the tangent plane to the hyperboloid  $x^2 + y^2 - z^2 = 18$  at  $(3, 5, -4)$ .

**solution.** Let  $f(x, y, z) = x^2 + y^2 - z^2$ , then its gradient at  $(3, 5, -4)$  is

$$\nabla f(3, 5, -4) = 2(x, y, -z)|_{(3, 5, -4)} = 2(3, 5, 4).$$

The tangent plane at  $(3, 5, -4)$  is

$$0 = (3, 5, 4) \cdot (x - 3, y - 5, z + 4) = 3(x - 3) + 5(y - 5) + 4(z + 4).$$

Or

$$3x + 5y + 4z = 18.$$

7. (10pts) A particle moves in such a way that its acceleration is constantly equal to  $-\mathbf{k}$ . If the position when  $t = 0$  is  $(0, 0, 1)$  and the velocity at  $t = 0$  is  $\mathbf{i} + \mathbf{j}$ , when and where does the particle fall below the plane  $z = 0$ ?

**solution.** Since  $\mathbf{v}(t) = \int \mathbf{a}(t) dt$  and  $\mathbf{c}(t) = \int \mathbf{v}(t) dt$ , we have

$$\mathbf{a}(t) = -\mathbf{k} = (0, 0, -1) \Rightarrow \mathbf{v}(t) = (c_1, c_2, -t + c_3)$$

$$\mathbf{v}(0) = \mathbf{i} + \mathbf{j} = (1, 1, 0) = (c_1, c_2, c_3) \Rightarrow \mathbf{v}(t) = (1, 1, -t)$$

$$\mathbf{v}(t) = (1, 1, -t) \Rightarrow \mathbf{c}(t) = (t + c_1, t + c_2, -t^2/2 + c_3)$$

$$\mathbf{c}(0) = (0, 0, 1) = (c_1, c_2, c_3) \Rightarrow \mathbf{c}(t) = (t, t, -t^2/2 + 1)$$

Let  $z(t) = 0$ , we have  $-t^2/2 + 1 = 0$ , i.e.,  $t = \sqrt{2}$ . Hence, the particle falls below the plane at  $(\sqrt{2}, \sqrt{2}, 0)$  when  $t = \sqrt{2}$ .

8. (10pts) Show that, at a local maximum or minimum of  $\|\mathbf{r}(t)\|$ , the vector  $\mathbf{r}'(t)$  is perpendicular to  $\mathbf{r}(t)$ . That is,  $\mathbf{r}'(t) \cdot \mathbf{r}(t) = 0$  for all  $t$ .

**proof.** Let  $f(t) = \|\mathbf{r}(t)\| = \sqrt{x^2(t) + y^2(t) + z^2(t)}$ , then

$$\begin{aligned} f'(t) &= \frac{1}{2} \left( x^2(t) + y^2(t) + z^2(t) \right)^{-1/2} 2(x(t)x'(t) + y(t)y'(t) + z(t)z'(t)) \\ &= \frac{1}{\|\mathbf{r}(t)\|} \mathbf{r}'(t) \cdot \mathbf{r}(t). \end{aligned}$$

At a local maximum or minimum of  $f(t) = \|\mathbf{r}(t)\|$ , we have

$$0 = f'(t) = \frac{1}{\|\mathbf{r}(t)\|} \mathbf{r}'(t) \cdot \mathbf{r}(t)$$

which implies

$$\mathbf{r}'(t) \cdot \mathbf{r}(t) = 0.$$

9. (10pts) The arc length of the path  $\mathbf{c}(t) = (\cos 3t, \sin 3t, 2t^{3/2})$  for  $0 \leq t \leq 2$  is  
 (1)  $6\sqrt{2}$ , (2)  $2(3\sqrt{3} - 1)$ , (3)  $3 - \sqrt{3}$ , (4)  $3(2\sqrt{2} - 1)$ , (5) None of the above.

**solution.** (2)

$$\mathbf{c}'(t) = 3(-\sin 3t, \cos 3t, t^{1/2}), \quad \|\mathbf{c}'(t)\| = 3\sqrt{1+t}$$

$$L = \int_0^2 \|\mathbf{c}'(t)\| dt = 3 \int_0^2 \sqrt{1+t} dt = 2(1+t)^{3/2} \Big|_0^2 = 2(3\sqrt{3} - 1).$$

10. (10pts) Let  $\mathbf{c}(t)$  be a given path,  $a \leq t \leq b$ . Let  $s = \alpha(t)$  be a new variable, where  $\alpha$  is a strictly increasing  $C^1$  function given on  $[a, b]$ . For each  $s$  in  $[\alpha(a), \alpha(b)]$ , there is a unique  $t$  with  $\alpha(t) = s$ . Define the function  $\mathbf{d}(s) : [\alpha(a), \alpha(b)] \rightarrow \mathbb{R}^3$  by  $\mathbf{d}(s) = \mathbf{c}(t)$ . Show that  $\mathbf{c}(t)$  and  $\mathbf{d}(s)$  have the same arc length.

**proof.** The arc lengths for paths  $\mathbf{c}(t)$  and  $\mathbf{d}(s)$  are

$$L_1 = \int_a^b \|\mathbf{c}'(t)\| dt \quad \text{and} \quad L_2 = \int_{\alpha(a)}^{\alpha(b)} \|\mathbf{d}'(s)\| ds,$$

respectively. By the change of variable:  $s = \alpha(t)$ , one has

$$ds = \alpha'(t)dt, \quad \mathbf{d}'(s) = \frac{d}{ds}(\mathbf{d}(\alpha(t))) \cdot \frac{dt}{ds} = \frac{1}{\alpha'(t)} \frac{d}{dt}(\mathbf{c}(t)) = \frac{1}{\alpha'(t)} \mathbf{c}'(t).$$

Since  $\alpha$  is a strictly increasing  $C^1$  function given on  $[a, b]$ , we have  $\alpha'(t) > 0$  which implies

$$\|\mathbf{d}'(s)\| = \|\mathbf{c}'(t)\| \frac{1}{|\alpha'(t)|} = \frac{1}{\alpha'(t)} \|\mathbf{c}'(t)\|.$$

Now, we have

$$L_2 = \int_{\alpha(a)}^{\alpha(b)} \|\mathbf{d}'(s)\| ds = \int_a^b \frac{1}{\alpha'(t)} \|\mathbf{c}'(t)\| \alpha'(t) dt = \int_a^b \|\mathbf{c}'(t)\| dt = L_1.$$

This completes the proof.