Name: ____________________
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Midterm 2– Math 362 (10/27/09)
SHOW ALL RELEVANT WORK!!!

1. (10pts) Show that the curve $\mathbf{c}(t) = (1/t^3,\, e^t,\, 1/t)$ is a flow line of the velocity vector field $\mathbf{F}(x, y, z) = (-3z^4, y, -z^2)$.

$$\vec{c}\,'(t) \overset{?}{=} \vec{F}(\vec{c}(t))$$

$$\left.\begin{aligned} \vec{c}\,'(t) &= (-3t^{-4}, e^t, -t^{-2}) \\ \vec{F}(\vec{c}(t)) &= (-3t^{-4}, e^t, -t^{-2}) \end{aligned}\right\} \Rightarrow \vec{c}\,'(t) = \vec{F}(\vec{c}(t))$$

2. (10pts) Calculate the divergence and curl of the vector field

$$\mathbf{F}(x, y, z) = (x, y + \cos x, z + e^{xy}).$$

$$\operatorname{div} \vec{F} = 1 + 1 + 1 = 3$$

$$\operatorname{curl} \vec{F} = \begin{vmatrix} \bar{i} & \bar{j} & k \\ \partial_x & \partial_y & \partial_z \\ x & y+\cos x & z+e^{xy} \end{vmatrix}$$

$$= \left(xe^{xy}, -ye^{xy}, -\sin x\right)$$

3. (10pts) Suppose that $\nabla \cdot \mathbf{F} = 0$ and $\nabla \cdot \mathbf{G} = 0$. Which of the following necessarily have zero divergence?

(a) $\mathbf{F} + \mathbf{G}$ $\qquad\qquad$ (b) $\mathbf{F} \times \mathbf{G}$.

(a) $\operatorname{div}(\vec{F} + \vec{G}) = \operatorname{div} \vec{F} + \operatorname{div} \vec{G} = 0$

(b) $\operatorname{div}(\vec{F} \times \vec{G}) = \vec{G} \cdot \operatorname{curl} \vec{F} - \vec{F} \cdot \operatorname{curl} \vec{G}$ not necessarily zero

e.g., $\vec{F} = (y, z, x)$, $\vec{G} = (z, x, y)$, $\vec{F} \times \vec{G} = (zy - x^2, zx - y^2, xy - z^2)$

$\operatorname{div}(\vec{F} \times \vec{G}) = -2(x + y + z) \neq 0$

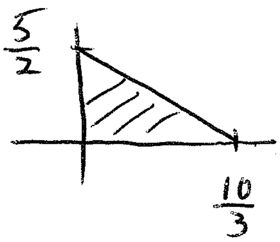
4. (10pts) Compute the volume of the solid bounded by the surface $z = \sin y$, the planes $x = 1$, $x = 0$, $y = 0$, and $y = \pi/2$, and the xy plane.

$$V = \int_0^1 \int_0^{\frac{\pi}{2}} \sin y \, dy \, dx$$

$$= -\cos y \Big|_0^{\frac{\pi}{2}} = 1$$

100	95	85	79	67	59	49	27
100	95	83	76	65	58	47	
100	94	81	72	65	58	46	
100	92	80	71	65	57	45	
100	91		71	64	54	43	
	91			61		41	
	90						
	90						
5	8	4	5	6	5	6	1 =40

5. (10pts) Let D be the region bounded by the positive x and y axes and the line $3x+4y=10$. Compute $\int\int_D (x^2+y^2)\,dA$.



$$\begin{cases} 0 \le x \le \frac{10}{3} \\ 0 \le y \le \frac{10-3x}{4} \end{cases}$$

$$\begin{cases} 0 \le x \le \frac{10-4y}{3} \\ 0 \le y \le \frac{5}{2} \end{cases}$$

$$\iint_D (x^2+y^2)\,dx\,dy = \int_0^{\frac{10}{3}} dx \int_0^{\frac{10-3x}{4}} (x^2+y^2)\,dy$$

$$= \int_0^{\frac{10}{3}} \left[\frac{10x^2-3x^3}{4} + \frac{1}{3}\left(\frac{10-3x}{4}\right)^3 \right] dy$$

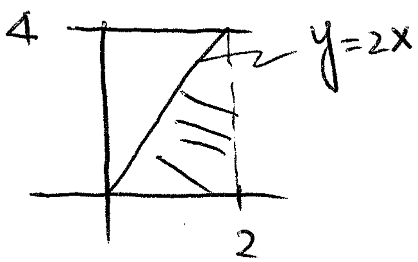
$$= \frac{1}{4}\left[\frac{10}{3}x^3 - \frac{3}{4}x^4\right]_0^{\frac{10}{3}} - \frac{1}{3}\cdot\frac{1}{4}\cdot\frac{4}{3}\left(\frac{10-3x}{4}\right)^4\Bigg|_0^{\frac{10}{3}}$$

$$= \frac{1}{4}\left[\left(\frac{10}{3}\right)^4 - \frac{3}{4}\left(\frac{10}{3}\right)^4\right] + \frac{1}{9}\left(\frac{10}{4}\right)^4$$

$$= \frac{1}{16}\left(\frac{10}{3}\right)^4 + \frac{1}{9}\left(\frac{5}{2}\right)^4$$

6. (10pts) Compute

$$\int_0^4 \int_{y/2}^2 e^{x^2}\,dx\,dy.$$



$$\begin{cases} 0 \le y \le 4 \\ \frac{y}{2} \le x \le 2 \end{cases}$$

$$\begin{cases} 0 \le x \le 2 \\ 0 \le y \le 2x \end{cases}$$

$$= \int_0^2 dx \int_0^{2x} e^{x^2}\,dy$$

$$= \int_0^2 2x e^{x^2}\,dx$$

$$= e^{x^2}\Big|_0^2 = e^4 - 1$$

7. (10pts) If $D = [-1,1] \times [-1,2]$, using the mean value theorem to show that

$$1 \leq \int\int_D \frac{1}{x^2+y^2+1}\,dxdy \leq 6.$$

Proof For $(x,y) \in [-1,1] \times [-1,2]$, i.e., $-1 \leq x \leq 1$, $-1 \leq y \leq 2$

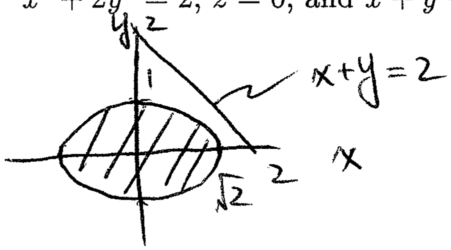
$$1 \leq x^2+y^2+1 \leq 6 \quad \text{and} \quad \text{area}(D) = 6$$

$$\Rightarrow \frac{1}{6} \leq \frac{1}{x^2+y^2+1} \leq 1$$

by the mean value thrm

$$1 = \frac{1}{6}\,\text{area}(D) \leq \iint_D \frac{1}{x^2+y^2+1}\,dxdy \leq \text{area}(D) = 6$$

8. (10pts) Set up (no calculation) ~~an~~ iterated integral for the volume of the solid bounded by $x^2+2y^2 = 2$, $z=0$, and $x+y+\cancel{z}\ 2z = 2$



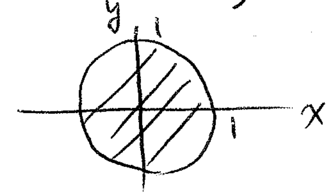
$$V = \iint_D \frac{2-x-y}{2}\,dxdy$$

$$= \int_{-1}^{1} dy \int_{-\sqrt{2-2y^2}}^{\sqrt{2-2y^2}} \frac{2-x-y}{2}\,dx$$

9. (10pts) Let $S^* = (0,1] \times [0, 2\pi)$ and define $T(r,\theta) = (r\cos\theta,\ r\sin\theta)$. Determine the image set S. Show that T is one-to-one on S^*.

$$S = \left\{ (x,y) = T(r,\theta) \;\middle|\; (r,\theta) \in (0,1] \times [0,2\pi) \right\}$$

$$= \left\{ (x,y) \;\middle|\; x^2 + y^2 \le 1 \right\}$$



Proof For $(r_1,\theta_1), (r_2,\theta_2) \in (0,1] \times [0,2\pi)$,

if $T(r_1,\theta_1) = T(r_2,\theta_2)$, i.e., $\begin{cases} r_1\cos\theta_1 = r_2\cos\theta_2 \\ r_1\sin\theta_1 = r_2\sin\theta_2 \end{cases}$

need to prove that $r_1 = r_2$ and $\theta_1 = \theta_2$.

By squaring and adding, we get $r_1^2 = r_2^2 \Rightarrow r_1 = r_2$

$\Rightarrow \cos\theta_1 = \cos\theta_2$ and $\sin\theta_1 = \sin\theta_2$

$\Downarrow$

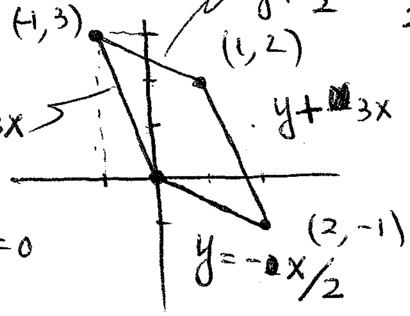
$\theta_1 = \theta_2$ or $\theta_2 = 2\pi - \theta_1$

if $\theta_2 = 2\pi - \theta_1$, then $\sin\theta_2 = \sin(2\pi - \theta_1) = -\sin\theta_1 \neq \sin\theta_1$

Hence, $\theta_1 = \theta_2$.

10. (10pts) Let D^* be the parallelogram with vertices at $(-1,3)$, $(0,0)$, $(2,-1)$, and $(1,2)$, and D be the square $[0,1]\times[0,1]$. Find a T such that D is the image set of D^* under T.

$$y+\frac{x}{2}=\frac{5}{2} \rightarrow \frac{2}{5}y+\frac{1}{5}x=1$$



$$y+3x=5 \rightarrow \frac{1}{5}y+\frac{3}{5}x=1$$

$$y=-3x \rightarrow y+3x=0 \rightarrow \frac{1}{5}y+\frac{3}{5}x=0$$

$$y+\frac{x}{2}=0 \rightarrow \frac{2}{5}y+\frac{1}{5}x=0$$

Let
$$\begin{cases} u=\frac{2}{5}y+\frac{1}{5}x \\ v=\frac{1}{5}y+\frac{3}{5}x \end{cases}$$

$$(u,v)=T(x,y)=\left(\frac{2}{5}y+\frac{1}{5}x,\ \frac{1}{5}y+\frac{3}{5}x\right)$$