Lecture Notes 2006 Fall

The Mathematical Theory of FEMs

Brenner & Scott

Chapter 0 Basic Concepts

§0.1 Weak Formulation of BVPs

<u>2-pts BVPs</u>
$$\begin{cases} -\dfrac{d^2u}{dx^2} = f & \text{in } (0,1) \\ u(0)=0, \quad u'(1)=0 \end{cases} \qquad (0.1.1)$$

<u>Weak Formulation</u> Find $u \in V = \left\{ v \in L^2(0,1) \;\middle|\; a(v,v) \equiv \int_0^1 v'v'\,dx < +\infty \text{ and } v(0)=0 \right\}$ s.t.

<u>Variational</u>
$$a(u,v) \equiv \int_0^1 u'v'\,dx = \int_0^1 f v\,dx \quad \forall v \in V. \qquad (0.1.3)$$

"$(0.1.1) \Rightarrow (0.1.3)$" Let u be solution, v — any (sufficiently smooth) function

$$(f,v) = \int_0^1 f v\,dx = \int_0^1 -u''v\,dx = \int_0^1 u'v'\,dx - u'v\Big|_0^1 = a(u,v) - u'(0)v(0)$$
$$= a(u,v) \quad \text{if } v(0)=0.$$

<u>Question</u> $u'(x) = ?$ $\Big\langle$ classic / ?

<u>Thrm (0.1.4)</u> $f \in C^0[0,1]$, $u \in C^2[0,1]$ satisfies $(0.1.3) \Longrightarrow u$ solves $(0.1.1)$

<u>Proof</u> ~~$v \in V \cap C^1[0,1]$~~ $(0.1.3) \xrightarrow[\text{by parts}]{\text{integration}} (f+u'', v) = 0 \quad \forall v \in V \cap C^1[0,1]$ and $\underline{\underline{v(1)=0}}$

$$(f+u'', v) = u'(1)v(1)$$

$$\Rightarrow f+u'' \equiv 0 \text{ in } (0,1) \quad \left[\begin{array}{l} f+u'' \in C^0[0,1] \\ \text{If not, } \exists\, [x_0,x_1] \subset [0,1] \text{ s.t. } f+u''>0 \text{ in } [x_0,x_1] \\ \text{choose } v = \begin{cases} (x-x_0)^2(x-x_1)^2 & x\in[x_0,x_1] \\ 0 & \text{otherwise} \end{cases} \end{array}\right. \Rightarrow \begin{cases} (f+u'',v) \neq 0 & \text{contradiction} \\ v(1)=0 \end{cases}$$

$$u'(1) = u'(1)v(1) = (f+u'',v) = 0$$

Take $v(x) = x \Rightarrow u'(1) = 0$

$u \in V \Rightarrow u(0) = 0$ #

<u>Remark</u>

$u(x)=0$	essential	Dirichlet
$u'(x)=0$	natural	Neumann

<u>Question</u> Well-posedness of (0.1.3) ?

<u>§0.2 Ritz-Galerkin Approximation</u>

$S \subset V$ — a finite dimensional subspace

<u>RG Approx.</u> Find $u_S \in S$ s.t.

$$a(u_S, v) = (f, v) \quad \forall v \in S \qquad (0.2.1)$$

~~Thrm (0.2.2) Given $f \in L^2[0,1]$, (0.2.1) has a unique solution.~~

~~Proof~~ $S = \text{span}\{\phi_i\}_{i=1}^n \Rightarrow u_S = \sum_j U_j \phi_j$

$$\Rightarrow \underset{n\times n}{K}\,\underset{n\times 1}{U} = \underset{n\times 1}{F}, \quad K = (K_{ij})_{n\times n} \quad K_{ij} = a(\phi_j, \phi_i)$$

$$U = (U_j)_{n\times 1}, \quad F = (F_i) \quad F_i = (f, \phi_i)$$

stiffness matrix

<u>Remark</u> (1) K is symmetric,

(2) K is positive definite, i.e., $\langle KV, V\rangle \geq 0 \quad \forall V \in R^n$
and "$\langle KV, V\rangle = 0 \Rightarrow V = 0$".

<u>Proof</u> $V = (V_j)$, let $v = \sum_j V_j \phi_j$

$$\Rightarrow \langle KV, V\rangle = V^t K V = a(v,v) \geq 0 \quad \forall v$$

$$\langle KV, V\rangle = 0 \Rightarrow a(v,v) = \int_0^1 (v')^2 dx = 0$$

$$\Rightarrow v' \equiv 0 \text{ in } (0,1)$$

$$\overset{?}{\Rightarrow} v = \text{constant} \xrightarrow{v(0)=0?} v \equiv 0 \text{ in } (0,1) \Rightarrow V = 0.$$

<u>Thrm(0.2.2)</u> <u>Given $f \in L^2[0,1]$</u>, (0.2.1) has a unique solution.

<u>Proof</u> (i) K is sym pos. def. $\Rightarrow$ $KU = F$ has a unique sol. ~~$\Rightarrow$ (0.2.1) has~~

~~(ii) a finite dimensional prob. is equivalent to that $KV = 0$ has only trivial sol.~~

§0.3 Error Estimates

<u>exact sol.</u> $a(u,v) = (f,v) \quad \forall v \in V$

<u>approx.</u> $a(u_S, v) = (f, v) \quad \forall v \in S \subset V$

$$\Longrightarrow a(u - u_S, v) = 0 \quad \forall v \in S$$

error equ. or orthogonal property

<u>the energy norm</u> $\|v\|_E = \sqrt{a(v,v)} = \sqrt{\int_0^1 (v')^2 dx}$

<u>the Schwarz inequality</u> $|a(u,v)| \leq \|u\|_E \|v\|_E \quad \forall u, v \in V.$

2nd Lecture Review

(1) 2-pts BVPs $\qquad (1) \Rightarrow (2)$ and $(2) \xRightarrow{\text{if (2) is smooth}} (1)$

(2) Variational Problem $\quad \left[\begin{array}{l} u \in V \\ a(u,v) = f(v) \quad \forall v \in V \end{array}\right.$

BCs: natural, essential

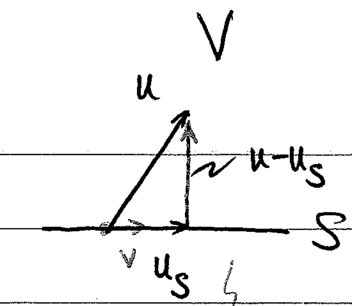
(3) R-G Approximation $\quad \left[\begin{array}{l} u_s \in S \subset V \\ a(u_s, v) = f(v) \quad \forall v \in S = \text{span}\{\phi_i\}_{i=1}^{n} \end{array}\right.$

system of AEs $\quad KU = F$

K is sym. pos. def.

Let $Y \in R^n$ and $v = Y_1\phi_1 + \cdots + Y_n\phi_n$

$$Y^t K Y = a(v, v)$$



$$a(u-u_S, v) = 0 \quad \forall v \in S$$

Thrm (0.3.3) $\quad \|u-u_S\|_E = \min_{v\in S} \|u-v\|_E$

$$\Updownarrow$$

~~Proof~~

Proof

$$\|u-u_S\|_E \le \|u-v\|_E \quad \forall v \in S.$$

$$\|u-u_S\|_E^2 = a(u-u_S, u-u_S) \overset{\text{orthogonality}}{=} a(u-u_S, u-v) \quad \forall v\in S$$

$$\le \|u-u_S\|_E \, \|u-v\|_E \quad \forall v \in S. \qquad \#$$

L^2-norm error estimate (a "duality" argument)

the dual problem: $\begin{cases} -w'' = u-u_S & \text{in } (0,1) \\ w(0) = w'(1) = 0 \end{cases} \Rightarrow a(w, v) = (u-u_S, v) \quad \forall v \in V$

$$\Rightarrow \|u-u_S\|^2 = a(w, u-u_S) = a(w-v, u-u_S) \quad \forall v \in S$$

$$\le \|w-v\|_E \, \|u-u_S\|_E$$

$$\Rightarrow \quad \|u-u_S\| \le \|u-u_S\|_E \frac{\|w-v\|_E}{\|u-u_S\|}$$

$$= \|u-u_S\|_E \frac{\|w-v\|_E}{\|w''\|}$$

Assume that $\inf_{v\in S} \|w-v\|_E \le \varepsilon \|w''\|$?

Thrm (0.3.5) $\quad \|u-u_S\| \le \varepsilon^2 \|u''\| = \varepsilon^2 \|f\|.$

Proof $\quad \|u-u_S\| \le \varepsilon \|u''\| \dfrac{\varepsilon \|w''\|}{\|w''\|} = \varepsilon^2 \|u''\|. \qquad \#$

S: (1) polynomials of degree n
(2) piecewise polynomials (spline, FE, etc.)
(3) trigonometric functions
(4) wavelet ...

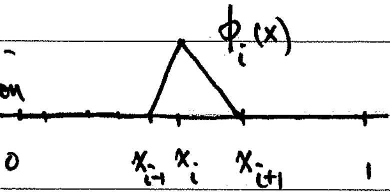
§0.4 Piecewise Polynomials Spaces – Finite Element Method

a partition : $0 = x_0 < x_1 < \cdots < x_n = 1$, $\quad I_i = [x_{i-1}, x_i]$ and $h_i = x_i - x_{i-1}$

$$S = \left\{ v \in C^0[0,1] \;\middle|\; v\big|_{I_i} \text{ is a linear poly. } (i=1,\cdots,n) \text{ and } v(0)=0 \right\} \subset V$$

$$\dim S = \underbrace{n}_{\#\text{ of intervals}} \cdot \overbrace{2}^{2 \text{ DoF at each interval}} - \underbrace{(n-1)}_{\#\text{ of interior nodes}} - \overbrace{1}^{\text{B.C.}} = n$$

<u>nodal basis function</u>



$\phi_i(x) \in S$ and $\phi_i(x_j) = \delta_{ij} = \begin{cases} 1 & x_j = x_i \\ 0 & x_j \neq x_i \end{cases}$

$$\Rightarrow \phi_i(x) = \begin{cases} \dfrac{x - x_{i-1}}{x_i - x_{i-1}} & x \in I_i \\[2ex] \dfrac{x - x_{i+1}}{x_i - x_{i+1}} & x \in I_{i+1} \\[2ex] 0 & x \notin I_i \cup I_{i+1} \end{cases}$$

<u>Properties</u>

(1) $\begin{cases} -\phi_i''(x) = 0 \quad \text{in } (x_{i-1}, x_i) \cup (x_i, x_{i+1}) \\ \phi_i(x_{i-1}) = 0, \; \phi_i(x_i) = 1, \; \phi_i(x_{i+1}) = 0 \end{cases}$

(2) $\{\phi_i(x)\}_{i=1}^{n}$ is linearly indep.

$$\sum_i c_i \phi_i(x) \equiv 0 \text{ in } [0,1] \Longrightarrow c_j = \sum_i c_i \phi_i(x_j) = 0.$$

$$\Longrightarrow S = \operatorname{span}\{\phi_i(x)\}_{i=1}^{n}$$

<u>interpolant</u> Given $v \in C^0[0,1]$, define the interpolant v_I of v as

$$\left.\begin{aligned} &(1)\ v_I \in S \\ &(2)\ v_I(x_i) = v(x_i) \end{aligned}\right\} \Longrightarrow v_I(x) = \sum_i v(x_i)\phi_i(x)$$

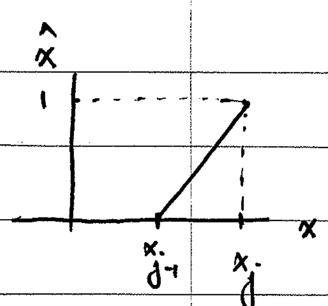
Thm (0.4.5) Let $h = \max_{1 \le i \le n} h_i$. Then for $u \in C^2[0,1] \cap V$

$$\|u - u_I\|_E \le C h \|u''\|.$$

Proof Let $e = u - u_I$,

$$\|e\|_E^2 \le C h^2 \|u''\|^2 \iff \sum_j \int_{x_{j-1}}^{x_j} e'(x)^2 dx \le \sum_j \int_{x_{j-1}}^{x_j} u''(x)^2 dx \cdot C h^2$$

$$\Leftarrow \int_{x_{j-1}}^{x_j} e'(x)^2 dx \le C (x_j - x_{j-1})^2 \int_{x_{j-1}}^{x_j} u''(x)^2 dx \overset{u_I''|_{I_j}=0}{=} C h_j^2 \int_{x_{j-1}}^{x_j} e''(x)^2 dx$$



$$\hat{x} : [x_{j-1}, x_j] \longrightarrow [0,1] \qquad \hat{x} = \frac{x - x_{j-1}}{x_j - x_{j-1}} \quad \text{or} \quad x = x_{j-1} + h_j \hat{x} : [0,1] \to [x_{j-1}, x_j]$$

$$\cancel{\hat e(\hat x) =}\; e(x) = e\left(x_{j-1} + h_j \hat{x}\right) = \hat{e}(\hat{x})$$

$$e'(x) = \frac{de(x)}{dx} = \frac{d\hat{e}(\hat{x})}{d\hat{x}} \cdot \frac{d\hat{x}}{dx} = \hat{e}'(\hat{x}) \frac{1}{h_j}$$

$$e''(x) = \frac{d^2 e(x)}{dx^2} = \frac{1}{h_j} \frac{d\hat{e}'(\hat{x})}{dx} = \frac{1}{h_j} \hat{e}''(\hat{x}) \frac{d\hat{x}}{dx} = \frac{1}{h_j^2} \hat{e}''(\hat{x}).$$

$$dx = h_j \, d\hat{x}$$

$$\Rightarrow \int_0^1 \hat{e}'(\hat{x})^2 \frac{1}{h_j^2} h_j \, d\hat{x} \le C h_j^2 \int_0^1 \hat{e}''(\hat{x})^2 \frac{1}{h_j^4} h_j \, d\hat{x}$$

$$\iff \int_0^1 \hat{e}'(\hat{x})^2 d\hat{x} \le C \int_0^1 \hat{e}''(\hat{x})^2 d\hat{x} \quad \text{(the scaling argument)}$$

$$\hat{e}(0) = \hat{e}(1) = 0 \xRightarrow{\text{Rolle's Thrm}} \exists\, \xi \in (0,1) \text{ s.t. } \hat{e}'(\xi) = 0$$

$$\Rightarrow \hat{e}'(\hat{x}) = \int_{\xi}^{\hat{x}} \hat{e}''(y)\, dy$$

$$|\hat{e}'(\hat{x})| = \left|\int_\xi^{\hat{x}} \hat{e}''(y)\,dy\right| \le \left|\int_\xi^{\hat{x}} 1^2\,dy\right|^{\frac{1}{2}} \left|\int_\xi^{\hat{x}} \hat{e}''(y)^2\,dy\right|^{\frac{1}{2}}$$

$$= |\hat{x}-\xi|^{\frac{1}{2}} \left|\int_\xi^{\hat{x}} \hat{e}''(y)^2\,dy\right|^{\frac{1}{2}} \le |\hat{x}-\xi|^{\frac{1}{2}} \left|\int_0^1 \hat{e}''(y)^2\,dy\right|^{\frac{1}{2}}$$

$$\int_0^1 \hat{e}'(\hat{x})^2\,d\hat{x} \le \int_0^1 |\hat{x}-\xi|\,d\hat{x} \int_0^1 \hat{e}''(y)^2\,dy$$

$$\le \sup_{\xi\in[0,1)} \int_0^1 |\hat{x}-\xi|\,d\hat{x} \cdot \int_0^1 \hat{e}''(\hat{x})^2\,d\hat{x} = \frac{1}{2}\int_0^1 \hat{e}''(\hat{x})^2\,d\hat{x}.$$

#

<u>Corollary</u> (0.4.7) $\|u-u_S\| + h\|u-u_S\|_E \le C h^2 \|u''\|$.

<u>Remark (0.4.8)</u> $\mathcal{I} : \overset{\circ}{C}[0,1] \to S$ by $\mathcal{I} v = v_I$ defines a linear operator and $\mathcal{I}$ is a projection ($\mathcal{I}^2 = \mathcal{I}$).

<u>§0.5 Relationship to Difference Methods</u>

For ϕ_i described above,

$$K = \begin{pmatrix} \ddots & \ddots & & 0 \\ & K_{i\,i-1} & K_{ii} & K_{i\,i+1} \\ 0 & & \ddots & \ddots \end{pmatrix} \qquad \begin{aligned} K_{ii} &= \frac{1}{h_i} + \frac{1}{h_{i+1}} \quad i=1,\cdots,n-1; \quad K_{nn} = \frac{1}{h_n} \\ K_{i\,i-1} &= -\frac{1}{h_i} = K_{i-1,i} \\ K_{i\,i+1} &= -\frac{1}{h_{i+1}} = K_{i+1,i} \end{aligned}$$

$$F_i = (f,\phi_i) = \int_{x_{i-1}}^{x_{i+1}} f\phi_i\,dx \qquad \text{if } f(x) = f(x_i) + f'(\xi_i)(x-x_i)$$

$$= f(x_i)\,\tfrac{1}{2}(h_i + h_{i+1}) + \tfrac{1}{2}(h_i+h_{i+1})\,O(h) \sim O(h^2) \text{ if } 1-\frac{h_i}{h_{i+1}} = O(h)$$

the i^{th} eq
$$-\frac{1}{(h_i+h_{i+1})/2}\left(\frac{U_{i+1}-U_i}{h_{i+1}}-\frac{U_i-U_{i-1}}{h_i}\right)=\frac{F_i}{(h_i+h_{i+1})/2}=f(x_i)+O(h)$$
$$\parallel$$
$$-\frac{d^2U(x_i)}{dx^2}+O(h)$$

$h_i = h_{i+1}$
$$-\frac{U_{i+1}-2U_i+U_{i-1}}{h^2}=f(x_i)+O(h^2)$$

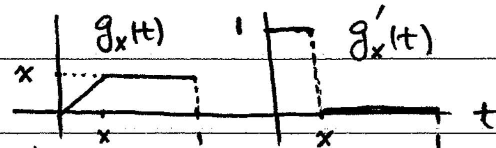
Remark For general ~~meth~~ mesh, it is 2^{nd}-order in both L^2 and L^∞ norms. even through the difference equations are formally consistent to 1^{st}-order.

Advantages systematic, easy handling BCs, math. profound, clean notation,

§0.6 Computer Implementation of FEMs

§0.7 Local Estimates

DEs $\begin{cases} -u''(x) = f(x) & x \in (0,1) \\ u(0) = 0,\ u'(0) = 0 \end{cases}$



Green's Function $g_x(t) = \begin{cases} t & t \leqslant x \\ x & t \geqslant x \end{cases}$ where x is a given pt in $[0,1]$

$$\begin{cases} -g_x''(t) = \delta(x) & \text{in } t \in (0,1) \\ g_x(0) = 0,\ g_x'(1) = 0 \end{cases}$$ general def. of Green's function

- For any $v \in V$, $\quad v(x) = a\big(v, g_x(t)\big)$

Proof $\displaystyle a(v, g_x) = \int_0^1 v'(t)\, g_x'(t)\, dt = \int_0^x + \int_x^1 v' g_x' \, dt$

$$= v(t)\, g_x'(t)\Big|_0^x - \int_0^x v(t)\, g_x''(t)\, dt + v g_x'\Big|_x^1 - \int_x^1 v\, g_x''\, dt$$

$$= v(x)\, g_x'(x^-) - v(0)\, g_x'(0) + v(1)\, g_x'(1) - v(x)\, g_x'(x^+)$$

$$= v(x)\Big(g_x'(x^-) - g_x'(x^+)\Big) = v(x)$$ #

$\Longrightarrow \quad u(x) - u_S(x) = a\big(u - u_S,\ g_x\big) = a\big(u - u_S,\ g_x - v\big) \quad \forall\, v \in S.$

Let $S = \text{span}\{\phi_i(x)\}_{i=1}^n$ defined in the previous section

$\Longrightarrow \quad g_{x_i}(t) \in S$

$\Longrightarrow \quad \big(u - u_S\big)(x_i) = a\big(u - u_S,\ g_x\big) = 0 \qquad \forall\, i = 1, \cdots, n$

$\Longrightarrow \quad \boxed{u_S(x) \equiv u_I(x)}$

Thrm $\|u-u_I\|_{max} \leq C h^2 \|u''\|_{max}$ $\qquad \|f\|_{max} = \max_{x\in[0,1]} |f(x)|$

Proof Let $e(x) = u(x) - u_I(x)$

On I_i $\quad e(x_{i-1}) = e(x_i) = 0$

$$\Rightarrow |e(x)| = \left| \int_{x_{i-1}}^{x} e'(y)\,dy \right| \leq \left(\int_{x_{i-1}}^{x} 1\,dy \right)^{\frac{1}{2}} \left(\int_{x_{i-1}}^{x} e'(y)^2\,dy \right)^{\frac{1}{2}}$$

$$\leq (x - x_{i-1})^{\frac{1}{2}} \left(\int_{x_{i-1}}^{x_i} e'(y)^2\,dy \right)^{\frac{1}{2}}$$

$$\leq h_i^{\frac{1}{2}} \left(\int_{x_{i-1}}^{x_i} e'(y)^2\,dy \right)^{\frac{1}{2}}$$

$$\leq h_i^{\frac{1}{2}}\, C h_i \left(\int_{x_{i-1}}^{x_i} u''(y)^2\,dy \right)^{\frac{1}{2}}$$

$$\leq C h_i^2 \max_{x\in I_i} |u''| \leq C h_i \|u''\|_{max}.$$

Or $\quad e(x) = \dfrac{u''(\xi_i)}{2!}(x-x_{i-1})(x-x_i) \quad \exists\, \xi_i \in (x_{i-1}, x_i)$

$$\max_{x\in I_i} |e(x)| \leq \max_{x\in I_i} \frac{1}{2}(x-x_{i-1})(x-x_i) \max_{x\in I_i} |u''|$$

$$= \frac{1}{8} h_i^2 \max_{x\in I_i} |u''(x)|$$

$$\Rightarrow \|e(x)\|_{max} \leq \frac{1}{8} h^2 \|u''\|_{max}.$$