

Chapter 1 Sobolev Spaces

§1.1 Review of Lebesgue Integration Theory

- domain Ω : a Lebesgue-measurable subset of $\mathbb{R}^n$ with non-empty interior
- function $f(x)$: a real-valued function defined on Ω that is L -measurable
- the L -integral $\int_{\Omega} f(x) dx$, where dx denotes L -measure

• norm

$$\|f\|_{L^p(\Omega)} = \begin{cases} \left(\int_{\Omega} |f(x)|^p dx \right)^{\frac{1}{p}} & 1 \leq p < +\infty \\ \text{ess sup } \{ |f(x)| : x \in \Omega \} & p = +\infty \end{cases}$$

- the Lebesgue Space

$$L^p(\Omega) = \left\{ f : \|f\|_{L^p(\Omega)} < +\infty \right\}$$

- $\forall f, g \in L^p, \quad "f = g \iff \|f - g\|_{L^p(\Omega)} = 0"$

- Minkowski's inequality $\forall f, g \in L^p(\Omega), 1 \leq p \leq \infty$

$$\|f + g\|_{L^p(\Omega)} \leq \|f\|_{L^p(\Omega)} + \|g\|_{L^p(\Omega)}$$

- Hölder's Inequality For $p, q \in [1, \infty]$ s.t. $\frac{1}{p} + \frac{1}{q} = 1$

$$f \in L^p(\Omega), g \in L^q(\Omega) \Rightarrow \begin{cases} fg \in L^1(\Omega), \\ \|fg\|_{L^1(\Omega)} \leq \|f\|_{L^p(\Omega)} \|g\|_{L^q(\Omega)} \end{cases}$$

- Schwarz' inequality $p=q=2$.

$$\|fg\|_{L^1(\Omega)} \leq \|f\|_{L^2(\Omega)} \|g\|_{L^2(\Omega)}$$

- $L^p(\Omega)$ is a linear (vector) space.

Proof $f, g \in L^p(\Omega)$ and $\alpha, \beta \in \mathbb{R}$

$$\xrightarrow{\text{M-ineq}} \alpha f + \beta g \in L^p(\Omega).$$

#

- Definition of norm Given a linear space V ,

$$\|\cdot\| : V \rightarrow \mathbb{R} \text{ is a norm} \iff \begin{cases} (1) \|v\| \geq 0 \quad \forall v \in V \\ \text{and } \|v\| = 0 \iff v = 0 \\ (2) \|cv\| = |c| \|v\| \quad \forall c \in \mathbb{R} \\ (3) \|v+w\| \leq \|v\| + \|w\| \\ \forall v, w \in V \end{cases}$$

$\|\cdot\|_{L^p(\Omega)}$ is a norm

- a normed vector space $(V, \|\cdot\|)$
- A normed vector space V is complete (Banach space)
- $\iff$ every Cauchy sequence has a limit in V
- $\{v_n\}_{n=1}^{\infty} : \|v_n - v_m\| \rightarrow 0 \text{ as } n, m \rightarrow \infty.$

- $L^p(\Omega)$ for $p \in [1, \infty]$ is a Banach space.

- inclusion if Ω is bounded

$$\Rightarrow L^p(\Omega) \subset L^q(\Omega) \text{ for } p \geq q.$$

Proof $f \in L^p(\Omega) \stackrel{?}{\implies} f \in L^q(\Omega)$

$$\begin{aligned} \|f\|_{L^q(\Omega)}^q &= \int_{\Omega} 1 \cdot |f|^q dx \leq \|1\|_{L^{r'}(\Omega)} \| |f|^q \|_{L^r(\Omega)} \quad \left[\frac{1}{r} + \frac{1}{r'} = 1 \right] \\ &= \left(\int_{\Omega} 1^{r'} dx \right)^{\frac{1}{r'}} \left(\int_{\Omega} |f|^{q \cdot r} dx \right)^{\frac{1}{r}} \end{aligned}$$

choose r such that $r q = p \stackrel{r \geq 1}{\implies} p \geq q.$

#

§1.2 Generalized (Weak) Derivatives

- point-wise definition

$$u'(x) = \lim_{h \rightarrow 0} \frac{u(x+h) - u(x)}{h}$$

- functions in L-spaces

- pointwise values are irrelevant
- determined by its global behavior.

- global definition the duality technique using $\mathcal{D}(\Omega)$

- multi-index $\alpha = (\alpha_1, \dots, \alpha_n)$ where $\alpha_i \geq 0$ are integers

$$|\alpha| = \sum_{i=1}^n \alpha_i \quad (\text{length})$$

- notations $\vec{x} = (x_1, \dots, x_n) \in \Omega \subset \mathbb{R}^n$

$$\vec{x}^\alpha = x_1^{\alpha_1} \cdot x_2^{\alpha_2} \cdot \dots \cdot x_n^{\alpha_n}, \quad \frac{\partial}{\partial \vec{x}} = \left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \right)$$

$$\left(\frac{\partial}{\partial \vec{x}} \right)^\alpha = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdot \dots \cdot \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}} = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdot \dots \cdot \partial x_n^{\alpha_n}}$$

$\phi \in C^\infty(\Omega) : \mathcal{D}^\alpha \phi, \mathcal{D}_x^\alpha \phi, \phi^{(\alpha)}, \partial_x^\alpha \phi \quad |\alpha| \text{-order derivative}$

- support $\text{sppt } u = \overline{\{x : u(x) \neq 0\}}$

- u has a "compact support" w.r.t. Ω

$$\iff \begin{cases} (1) \text{ spt } u \text{ is bounded} \\ (2) \text{ spt } u \subset \text{interior of } \Omega \end{cases}$$

Definition (1.2.1) Let Ω be a domain in $\mathbb{R}^n$

$$\mathcal{D}(\Omega) = C_0^\infty(\Omega) = \left\{ u \in C^\infty(\Omega) \mid u \text{ has a compact support in } \Omega \right\}$$

Ex. (1.2.2) $\mathcal{D}(\Omega)$ is not empty.

Assume that Ω contains the closed unit ball $B_1 = \{x \mid |x| \leq 1\}$

$$\Rightarrow \phi(x) = \begin{cases} \exp \left\{ \frac{1}{|x|^2 - 1} \right\}, & \text{if } |x| < 1 \\ 0, & \text{if } |x| \geq 1 \end{cases} \in \mathcal{D}(\Omega)$$

(1) $|x| < 1$ $\phi^{(\alpha)}(x) = P_\alpha(x) e^{-t} t^{|x|+k}$ for some poly. P_α

Proof $|x|=0$, $\phi(x) = e^{-t}$

$$\begin{aligned} \underline{|x|=1}, \quad \partial_{x_i} \phi(x) &= -e^{-t} \cdot \frac{\partial t}{\partial x_i} \\ &= (-2x_i) e^{-t} t^{|x|+1} \end{aligned}$$

$$\begin{aligned} \text{and } t &= \frac{1}{1-|x|^2} = \frac{1}{1-\sum_j x_j^2} \\ \frac{\partial t}{\partial x_i} &= - \left(\frac{1}{1-\sum_j x_j^2} \right)^2 \cdot (-2x_i) \\ &= 2x_i t^2 \end{aligned}$$

proof by induction

$$|x| \rightarrow 1 \iff t \rightarrow \infty$$

$$(2) \underline{|x| > 1} \quad \phi^{(\alpha)}(x) = 0$$

$$(3) \underline{|x| = 1} \quad \lim_{|x| \rightarrow 1} \phi^{(\alpha)}(x) = \lim_{|x| \rightarrow 1} \frac{P_\alpha(x) t^{|x|+k}}{e^t} \rightarrow 0$$

Definition (1.2.3) Given a domain $\Omega \subset \mathbb{R}^n$

locally integrable functions

$$L^1_{loc}(\Omega) = \left\{ f \mid f \in L^1(K) \quad \forall K \subset \text{interior } \Omega \right\}$$

$$\bullet \quad C^0(\Omega) \subset L^1_{loc}(\Omega)$$

• functions in $L^1_{loc}(\Omega)$ can behave badly near $\partial\Omega$

$$\text{e.g., } \exp \left\{ \exp \frac{1}{\text{dist}(x, \partial\Omega)} \right\}$$

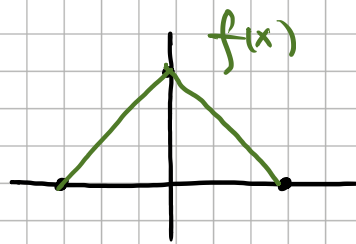
Definition (1.2.4)

$f \in L^1_{loc}(\Omega)$ has a weak derivative $\iff \exists g \in L^1_{loc}(\Omega) \text{ s.t.}$

$$g = D_w^\alpha f$$

$$\int_\Omega g \phi dx = (-1)^{|\alpha|} \int_\Omega f \phi^{(\alpha)} dx, \quad \forall \phi \in \mathcal{D}(\Omega)$$

Ex. $f(x) = 1 - |x|, \quad x \in \Omega = [-1, 1]$



$$\begin{aligned}
 \int_{-1}^1 f \phi' dx &= \int_{-1}^0 f \phi' dx + \int_0^1 f \phi' dx \\
 &= - \int_{-1}^0 f' \phi dx + [f \phi]_{-1}^0 - \int_0^1 f' \phi dx + [f \phi]_0^1 \\
 &= - \left[\int_{-1}^0 (1) \phi dx + \int_0^1 (-1) \phi dx \right] \Rightarrow D'_\omega f = \begin{cases} 1, & x < 1 \\ -1, & x > 1 \end{cases}
 \end{aligned}$$

$$\begin{aligned}
 \int_{-1}^1 f \phi'' dx &= \int_{-1}^0 f \phi'' dx + \int_0^1 f \phi'' dx \\
 &= - \left[\int_{-1}^0 f' \phi' dx + \int_0^1 f' \phi' dx \right] + [f \phi'']_{-1}^0 + [f \phi'']_0^1 \\
 &= \left[\int_{-1}^0 f'' \phi dx + \int_0^1 f'' \phi dx \right] - \left\{ [f' \phi']_{-1}^0 + [f' \phi']_0^1 \right\} \\
 &\quad - [f'(0^-) - f'(0^+)] \phi'(0) \neq 0
 \end{aligned}$$

$\Rightarrow D''_\omega f$ DNE.

Remark of this example

(1) $D^\alpha_\omega f = D^\alpha f$ at regular pt.

(2) $f \in C^0(\Omega) \Rightarrow D'_\omega f$ exists (depending on the dimension n)

Ex. (1.2.6) $f(x) = |x|^\beta$ $\bar{n}$ defined on $\Omega = \{\vec{x} \in \mathbb{R}^n \mid |x| < 1\}$.

$\forall \varphi \in \mathcal{D}(\Omega)$, for $|\alpha| = 1$, let $\alpha = (0, \dots, \overset{k}{1}, \dots, 0)$

$$\begin{aligned}
 \int_\Omega f(x) \varphi^{(\alpha)}(x) dx &= - \int_\Omega \frac{\partial}{\partial x_k} f(x) \varphi(x) dx & \frac{\partial}{\partial x_k} |x|^\beta &= \beta |x|^{\beta-1} \frac{\partial |x|}{\partial x_k} \\
 &= -\beta \int_\Omega |x|^{\beta-1} \frac{x^\alpha}{|x|} \varphi(x) dx < +\infty & &= \beta |x|^{\beta-1} \frac{x^\alpha}{|x|}
 \end{aligned}$$

If $\int_0^1 r^{\beta-1} \cdot r^{n-1} dr < +\infty$

$\int_0^1 r^{\beta+n-2} dr$

$\begin{cases} E, & \beta > 1-n \\ \text{DEN}, & \beta < 1-n \end{cases}$

$\Rightarrow D^\alpha_\omega |x|^\beta = \beta |x|^{\beta-1} \frac{x^\alpha}{|x|}$

Proposition $\varphi \in C^{|\alpha|}(\Omega) \Rightarrow D_{\omega}^{\alpha} \varphi$ exists and $D_{\omega}^{\alpha} \varphi = \tilde{D}^{\alpha} \varphi$.

HW #1, 3, 11, 13

§1.3 Sobolev Norms and Associated Spaces

• assumptions

$k \geq 0$ integer, $f \in L^1_{loc}(\Omega)$, $D_{\omega}^{\alpha} f$ exist $\forall |\alpha| \leq k$.

$|\alpha| = 0, 1, \dots, k$

• Sobolev norms

$$\|f\|_{k,p,\Omega} = \|f\|_{W_p^k(\Omega)} = \begin{cases} \left(\sum_{|\alpha| \leq k} \|D_{\omega}^{\alpha} f\|_{L^p(\Omega)}^p \right)^{\frac{1}{p}}, & p \in [1, \infty) \\ \max_{|\alpha| \leq k} \|D_{\omega}^{\alpha} f\|_{L^{\infty}(\Omega)}, & p = \infty \end{cases}$$

• Sobolev spaces

$$W_p^k(\Omega) = \left\{ f \in L^1_{loc}(\Omega) \mid \|f\|_{W_p^k(\Omega)} < +\infty \right\}$$

Lipschitz function

Proposition (1) $W_{\infty}^1(\Omega) = \text{Lip}(\Omega) \equiv \left\{ f \in L^{\infty}(\Omega) \mid \|f\|_{\text{Lip}(\Omega)} < +\infty \right\}$

where $\|f\|_{\text{Lip}(\Omega)} = \|f\|_{\infty} + \sup_{x \neq y \in \Omega} |f(x) - f(y)| / |x - y|$

$$(2) W_{\infty}^k(\Omega) = \left\{ f \in C^{k-1}(\Omega) \mid f^{(\alpha)} \in \text{Lip}(\Omega) \forall |\alpha| \leq k-1 \right\}$$

proof. see Ex. 1.x.14-15.

$$(3) W_1^1(\Omega) = \{ \text{absolute continuous functions on } \Omega \} \text{ when } \Omega \subset \mathbb{R}$$

Theorem $W_p^k(\Omega)$ is a Banach space.

Proof (1) $W_p^k(\Omega)$ is a normed vector space

(2) $W_p^k(\Omega)$ is complete.

Proof of (2) Let $\{v_j\}_{j=1}^{\infty}$ be a Cauchy sequence in $W_p^k(\Omega)$.

$$\text{i.e., } \|v_j - v_i\|_{W_p^k(\Omega)}^p \rightarrow 0 \text{ as } i, j \rightarrow \infty$$

$$\sum_{|\alpha| \leq k} \|D_w^{\alpha}(v_j - v_i)\|_{L^p(\Omega)}^p$$

$$\Rightarrow \{D_w^{\alpha} v_j\} \text{ is a C-sequence in } L^p(\Omega) \forall |\alpha| \leq k$$

$$\xrightarrow{L^p(\Omega) \text{ is complete}} \exists v^{\alpha} \in L^p(\Omega) \text{ s.t. } \|D_w^{\alpha} v_j - v^{\alpha}\|_{L^p(\Omega)} \rightarrow 0 \forall |\alpha| \leq k$$

$$\xrightarrow{\alpha=(0,\dots,0)} \lim_{j \rightarrow \infty} v_j = v = v^{(0,\dots,0)}$$

? $\Downarrow$?

$$L^p(\Omega) \ni v^{\alpha} \stackrel{?}{=} D_w^{\alpha} v$$

$$\int_{\Omega} (D_w^{\alpha} v_j - v^{\alpha}) \varphi \, dx \rightarrow 0 \quad \forall \varphi \in \mathcal{D}(\Omega)$$

$$\left| \int_{\Omega} (D_{\omega}^{\alpha} v_j - v^{\alpha}) g \, dx \right| \leq \|D_{\omega}^{\alpha} v_j - v^{\alpha}\|_{L^p(\Omega)} \|g\|_{L^q(\Omega)} \rightarrow 0$$

$$\begin{aligned} \int_{\Omega} v^{\alpha} g \, dx &= \lim_{j \rightarrow \infty} \int_{\Omega} D_{\omega}^{\alpha} v_j g \, dx = \lim_{j \rightarrow \infty} (-1)^{|\alpha|} \int_{\Omega} v_j g^{(\alpha)} \, dx \\ &= (-1)^{|\alpha|} \int_{\Omega} \left(\lim_{j \rightarrow \infty} v_j \right) g^{(\alpha)} \, dx = (-1)^{|\alpha|} \int_{\Omega} v g^{(\alpha)} \, dx \end{aligned}$$

$$\Rightarrow v^{\alpha} = v^{(\alpha_1, \alpha_2, \dots, \alpha_n)} = D_{\omega}^{\alpha} v \in L^p(\Omega)$$

$$\Rightarrow v \in W_p^k(\Omega). \quad \#$$

• a different definition

$$\overline{C^k(\Omega)}^{\|\cdot\|_{W_p^k(\Omega)}} = \begin{cases} W_p^k(\Omega), & p \in [1, \infty) \\ C^k(\Omega) \neq W_{\infty}^k(\Omega), & p = \infty \end{cases}$$

Theorem For any open set $\Omega \subset \mathbb{R}^n$

$$\Rightarrow \boxed{C^{\infty}(\Omega) \cap W_p^k(\Omega) \text{ is dense in } W_p^k(\Omega) \text{ for all } p < +\infty.}$$

Proof ($\Omega = \mathbb{R}^n$)

$$\forall f \in L^p(\Omega) = W_p^0(\Omega)$$

$\xi(x)$: mollification of $f(x)$

$$\text{Let } f_{\varepsilon}(x) = \int_{\Omega} f(y) \xi^{\varepsilon}(x-y) \, dy = f * \xi^{\varepsilon}(x)$$

where $\varphi^\varepsilon(x-y) = \varepsilon^{-n} \varphi\left(\frac{x-y}{\varepsilon}\right)$, $\varphi \in \mathcal{D}(\Omega)$ and $\int_{\Omega} \varphi = 1$.

By the dominated convergence theorem,

" $f_j(x) \rightarrow f(x)$ a.e. in Ω

and $|f_j(x)| \leq g(x) \forall j, x \in \Omega, g \in L^1(\Omega)$

$$\Rightarrow \begin{cases} \int_{\Omega} |f_j - f| dx \rightarrow 0 \\ \text{and } \int_{\Omega} f_j - \int_{\Omega} f \rightarrow 0 \end{cases}$$

(1) $f_\varepsilon \in C^\infty(\Omega)$

(2) " $f \in L^p(\Omega) \Rightarrow f_\varepsilon \rightarrow f$ as $\varepsilon \rightarrow 0$ in $L^p(\Omega)$ "

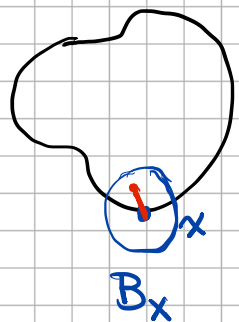
(3) " $f \in W_k^p(\Omega) \Rightarrow f_\varepsilon \rightarrow f$ as $\varepsilon \rightarrow 0$ in $W_k^p(\Omega)$ ". #

Remark $C^\infty(\bar{\Omega})$ is dense in $W_p^k(\Omega)$

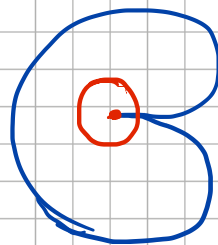
provided that Ω satisfies the segment condition:

$\forall x \in \partial\Omega, \exists$ an open ball B_x containing x and $\vec{n}_x \neq \vec{0}$ s.t.

$\forall z \in \bar{\Omega} \cap B_x$, the segment $\{\vec{z} + t\vec{n}_x \mid t \in (0,1)\} \subset \Omega$.



yes



not

• Sobolev semi-norm $k \geq 0$ integer, $f \in W_p^k(\Omega)$

$$|f|_{W_p^k(\Omega)} = \begin{cases} \left(\sum_{|\alpha|=k} \|D_w^\alpha f\|_{L^p(\Omega)}^p \right)^{\frac{1}{p}}, & p \in [1, \infty) \\ \max_{|\alpha|=k} \|D_w^\alpha f\|_{L^\infty(\Omega)} & p = \infty \end{cases}$$

§1.4 Inclusion Relations and Sobolev's Inequalities

Proposition (1.4.1) For any domain $\Omega \subset \mathbb{R}^n$,
 $0 \leq k \leq m$ are integers, $\forall p \in [1, \infty]$

$$\Rightarrow W_p^m(\Omega) \subset W_p^k(\Omega)$$

Proposition (1.4.2) $\Omega \subset \mathbb{R}^n$ is bounded, $k \geq 0$ integer, $1 \leq p \leq q \leq \infty$

$$\Rightarrow W_q^k(\Omega) \subset W_p^k(\Omega)$$

Remark $\exists k < m$ and $p > q$ s.t.

$$W_q^m(\Omega) \subset W_p^k(\Omega).$$

Theorem (1.4.5) Assume that Ω has a Lipschitz boundary.

$\Rightarrow \forall$ integer $k \geq 0$, $\forall p \in [1, \infty]$,

$\exists$ an extension mapping $E: W_p^k(\Omega) \rightarrow W_p^k(\mathbb{R}^n)$ s.t.

$$Ev|_{\Omega} = v \quad \forall v \in W_p^k(\Omega) \text{ and } \|v\|_{W_p^k(\mathbb{R}^n)} \leq C \|v\|_{W_p^k(\Omega)}.$$

Theorem (1.4.6) (Sobolev's Inequality)

Assume that $\Omega \subset \mathbb{R}^n$ has a L-boundary

and that integer $k > 0$ and real number $p \in [1, \infty)$ satisfy

$$\begin{cases} k \geq n, & \text{when } p=1 \end{cases}$$

$$\begin{cases} k > \frac{n}{p}, & \text{when } p > 1. \end{cases}$$

$$\boxed{k=1} \quad \underline{n=2}$$

$$1 > \frac{2}{p} \quad \text{with } \frac{1}{p} \text{ circled, } p > 2$$

$\Rightarrow \forall u \in W_p^k(\Omega)$, $\exists$ a constant $C > 0$ s.t.

$$\|u\|_{L^\infty(\Omega)} \leq C \|u\|_{W_p^k(\Omega)}.$$

[continuous imbedding: $\underline{L^\infty(\Omega)} \hookrightarrow W_p^k(\Omega)$: $L^\infty(\Omega) \subset W_p^k(\Omega)$
 $\|u\|_{L^\infty(\Omega)} \leq C \|u\|_{W_p^k(\Omega)}$]

Moreover, $\exists$ a continuous function in the $L^\infty(\Omega)$ equivalent class of u .

Corollary (1.4.7) $\Omega \subset \mathbb{R}^n$ has a L -boundary

Assume that integers $k > m > 0$ and $p \in [1, \infty)$ satisfy

$$\begin{cases} k-m \geq n, & \text{when } p=1 \\ k-m > \frac{n}{p}, & \text{when } p>1 \end{cases}$$

$\Rightarrow \forall u \in W_p^k(\Omega), \exists \text{ const. } c > 0 \text{ s.t.}$

$$\|u\|_{W_{\text{ob}}^{(m)}(\Omega)} \leq c \|u\|_{W_p^k(\Omega)}.$$

Moreover, $\exists$ a $C^m(\Omega)$ function in the $L^p(\Omega)$ equivalence class of u .

§1.5 Review of Chapter 0

• solution space $V = \{v \in W_2^1(\Omega) \mid v(0)=0\}, \Omega=(0,1)$

• Sobolev's inequality $k=1 > \frac{1}{2} = \frac{n}{p} \Rightarrow$ pointwise values are well

$$L^\infty(\Omega) \hookrightarrow W_2^1(\Omega)$$

$$L^\infty(\Omega) \hookrightarrow W_1^1(\Omega) \quad (k=1 \geq n=1)$$

- $S = \{v \in C^0(\Omega) \mid v|_{I_i} \in P_1(I_i)\}$ continuous piecewise linear
 $D_w' v$ is piecewise constant

$$\Rightarrow S \subset W_\infty'(\Omega) \subset W_2'(\Omega)$$

- (0.3.4) $\exists \varepsilon > 0$ s.t. $\inf_{v \in S} \|w - v\|_E \leq \varepsilon |w|_{W_2'(\Omega)} = \varepsilon \|w''\|_{L^2(\Omega)}$

$$a(v, v) = \int_0^1 (v')^2 dx = 0 \Rightarrow v(x) \equiv 0$$

Proof

$$\text{Poincaré's inequality: } \|v\|_{L^2(\Omega)}^2 \leq C a(v, v), \forall v \in V$$

$$\text{using } v(x) = \int_0^x D_w' v(s) ds \quad (\text{Ex \#16})$$

§1.6 Trace Theorems

domain $\Omega \subset \mathbb{R}^n$ and boundary $\partial\Omega$ is a manifold in $\mathbb{R}^{n-1}$

Theorem (1.6.6) Assume that Ω has a L-boundary and $p \in [1, \infty]$

$$\Rightarrow \exists \text{ a constant s.t. } \forall u \in W_p'(\Omega)$$

$$\|u\|_{L^p(\partial\Omega)} \leq C \|u\|_{L^p(\Omega)}^{1-\frac{1}{p}} \|u\|_{W_p'(\Omega)}^{\frac{1}{p}}.$$

Notations

$$W_p^{0,k}(\Omega) = \left\{ v \in W_p^k(\Omega) \mid v^{(\alpha)}|_{\partial\Omega} = 0 \text{ in } L^p(\partial\Omega) \quad \forall |\alpha| \leq k \right\}$$

Proof of Theorem in the case that $p=2$ and $\Omega \subset \mathbb{R}^2$ is a unit disk

$$\forall u \in W_2'(\Omega) \Rightarrow u|_{\partial\Omega} \in L^2(\partial\Omega) \text{ and } \|u\|_{L^2(\partial\Omega)} \lesssim \|u\|_{L^2(\Omega)}^{\frac{1}{2}} \|u\|_{W_2'(\Omega)}^{\frac{1}{2}}$$

Proof (1) $u \in C^1(\bar{\Omega})$, $\Omega = \{(r, \theta) \mid 0 \leq r < 1, \theta \in [0, 2\pi]\}$

$$u^2|_{\partial\Omega} = u^2(1, \theta) = \int_0^1 \partial_r (r^2 u^2(r, \theta)) dr = 2 \int_0^1 \left(u \nabla u \cdot \begin{pmatrix} x \\ y \end{pmatrix} + r u^2 \right) r dr$$

$$\leq 2 \int_0^1 \left(|u| |\nabla u| + u^2 \right) r dr$$

$$\Rightarrow \int_{\partial\Omega} u^2 ds = \int_0^{2\pi} u^2(1, \theta) d\theta \leq 2 \int_{\Omega} \left(|u| |\nabla u| + u^2 \right) dx dy$$

$$\leq 2\sqrt{2} \|u\|_{L^2(\Omega)} \|u\|_{W_2'(\Omega)}$$

(2) $u \in W_2'(\Omega)$ $C^1(\bar{\Omega})$ is dense in $W_2'(\Omega)$

$$\Rightarrow \exists \text{ a sequence } \{u_j\} \subset C^1(\bar{\Omega}) \text{ s.t. } \|u - u_j\|_{W_2'(\Omega)} \rightarrow 0$$

(a) $\{u_j\}$ is a C -sequence in $L^2(\partial\Omega)$

$$\begin{aligned}
 \forall k, j, \quad \|u_k - u_j\|_{L^2(\partial\Omega)} &\lesssim \|u_k - u_j\|_{L^2(\Omega)}^{\frac{1}{2}} \|u_k - u_j\|_{W_2^1(\Omega)}^{\frac{1}{2}} \\
 &\leq \|u_k - u_j\|_{W_2^1(\Omega)} \\
 &\leq \|u_k - u\|_{W_2^1(\Omega)} + \|u - u_j\|_{W_2^1(\Omega)} \rightarrow 0
 \end{aligned}$$

(a) $\xRightarrow{L^2(\partial\Omega) \text{ is complete}} \exists v \in L^2(\partial\Omega) \text{ s.t. } \|v - u_j\|_{L^2(\partial\Omega)} \rightarrow 0$

(b) v is independent of the choice of sequences.

Assume that $\{v_j\} \neq \{u_j\}$ in $C^1(\bar{\Omega})$ and that $\|u - v_j\|_{W_2^1(\Omega)} \rightarrow 0$

$$\begin{aligned}
 \|v - v_j\|_{L^2(\partial\Omega)} &\leq \|v - u_j\|_{L^2(\partial\Omega)} + \|u_j - v_j\|_{L^2(\partial\Omega)} \\
 &\lesssim \|v - u_j\|_{L^2(\partial\Omega)} + \|u_j - v_j\|_{W_2^1(\Omega)} \\
 &\leq \|v - u_j\|_{L^2(\partial\Omega)} + \|u_j - u\|_{W_2^1(\Omega)} + \|u - v_j\|_{W_2^1(\Omega)} \rightarrow 0
 \end{aligned}$$

$$\Rightarrow \|u_j\|_{L^2(\partial\Omega)} \lesssim \|u_j\|_{L^2(\Omega)}^{\frac{1}{2}} \|u_j\|_{W_2^1(\Omega)}^{\frac{1}{2}}$$

$$\xRightarrow{\lim_{j \rightarrow \infty}} \underset{u|_{\partial\Omega}}{\|v\|_{L^2(\partial\Omega)}} \lesssim \|u\|_{L^2(\Omega)}^{\frac{1}{2}} \|u\|_{W_2^1(\Omega)}^{\frac{1}{2}}$$

#

§1.7 Negative Norms and Duality

For $k < 0$, $p \in [1, \infty]$, let q satisfy $\frac{1}{p} + \frac{1}{q} = 1$

$$W_p^k(\Omega) = \left(W_q^{-k}(\Omega) \right)'$$