The Mathematical Theory of FEMs   Brenner & Scott

2006 Fall

Chapter 2 Variational Formulations of Elliptic BVPs

functional analysis tools.

§2.1 Inner-Product Spaces

V — linear space

Def (2.1.1) $b : V \times V \longrightarrow R$ is a bilinear form : $\begin{cases} v \mapsto b(v,w) \\ w \mapsto b(v,w) \end{cases}$ are linear on V.

symmetric : $b(v,w) = b(w,v) \ \forall v, w \in V$

$(\cdot,\cdot) : V \times V \to R$ is a (real) inner product

$\Leftrightarrow$ (1) $(\cdot,\cdot)$ is a sym bilinear form

(2) $(v,v) \geq 0 \ \forall v \in V$

(3) $(v,v) = 0 \Leftrightarrow v = 0$.

Def. (2.1.2) inner-product space $(V, (\cdot,\cdot))$

V is a linear space and $(\cdot,\cdot)$ is an inner-product defined on V

Examples (1) $V = R^n$, $(x,y) = \sum_{i=1}^{n} x_i y_i$

(2) $V = L^2(\Omega)$, $\Omega \subset R^n$, $(u,v)_{L^2(\Omega)} = \int_\Omega uv\,dx$

(3) $V = W_2^k(\Omega) = H^k(\Omega)$, $\Omega \subset R^n$, $(u,v)_k = \sum_{|\alpha| \leq k} (D^\alpha u, D^\alpha v)$.

HWs #4, 5, 6, 9, 10, 11, 12, 15

<u>Thrm (2.1.4)</u> (The Schwarz Inequality) $(V, (\cdot,\cdot))$ is an inner-product space

$$\Rightarrow_{(1)} \left|(u,v)\right| \leq (u,u)^{\frac{1}{2}}(v,v)^{\frac{1}{2}}$$

(2) $\left|(u,v)\right| = (u,u)^{\frac{1}{2}}(v,v)^{\frac{1}{2}} \Leftrightarrow$ u and v are linearly dependent.

<u>Proof</u> (1) <u>3 things</u>: $(u,u),\ (v,v),\ (u,v)$

<u>connection</u> $(u,u)+(v,v) \pm 2(u,v) = (u \pm v, u \pm v) \geq 0$

$$\Rightarrow -\tfrac{1}{2}[(u,u)+(v,v)] \leq (u,v) \leq \frac{1}{2}\left[(u,u)+(v,v)\right] \not\Rightarrow (1)$$

<u>modification</u> $(u,u)+(tv,tv) \pm 2(u,tv) = (u \pm tv, u \pm tv) \geq 0$

$$\Leftrightarrow \left|(u,v)\right| \leq \frac{1}{2}\left[\frac{(u,u)}{t} + t(v,v)\right]$$

$$= (u,u)^{\frac{1}{2}}(v,v)^{\frac{1}{2}} \quad \text{choosing } t = \frac{(u,u)^{\frac{1}{2}}}{(v,v)^{\frac{1}{2}}}$$

$$0 \leq (u+tv, u+tv) \Rightarrow (u,v) \geq -(u,u)^{\frac{1}{2}}(v,v)^{\frac{1}{2}}.$$

(2) "$\Leftarrow$" $u = \alpha v \Rightarrow \left|(u,v)\right| = (u,u)^{\frac{1}{2}}(v,v)^{\frac{1}{2}}.$

"$\Rightarrow$" from the proof of (1), $(u,v) = (u,u)^{\frac{1}{2}}(v,v)^{\frac{1}{2}}$ $\cancel{\not\Rightarrow 0 \neq (u+v, u+v)}$

$$\Leftrightarrow 0 = (u - tv, u - tv) \text{ with } t = \sqrt{\frac{(u,u)}{(v,v)}} \Rightarrow u - tv = 0$$

#

<u>Remark</u> Property "$(v,v)=0 \Leftrightarrow v=0$" is not used to prove the S-ineq.

$\Rightarrow \left|a(u,v)\right| \leq a(u,u)^{\frac{1}{2}}a(v,v)^{\frac{1}{2}}$ for $a(u,v) = \int_\Omega \nabla u \cdot \nabla v\, dx$ which is not an inner-product.

<u>Propersition (2.1.9)</u> $\|v\| = \sqrt{(v,v)}$ defines a norm in $(V, (\cdot,\cdot))$.

<u>Proof</u> (1) $\|\alpha v\| = |\alpha| \|v\|$; (2) $\|v\| = 0 \Leftrightarrow v = 0$;

(3) $\|u+v\| \le \|u\| + \|v\| \iff \|u+v\|^2 \le (\|u\| + \|v\|)^2 = \|u\|^2 + \|v\|^2 + 2\|u\|\|v\|$. #

$\ge \|u\|^2 + \|v\|^2 + 2(u,v) = (u+v, u+v)$

the S-ineq $\Longrightarrow (u,v) = \|u\| \|v\| \cos\theta$.

§2.2 Hilbert Spaces

<u>Def (2.2.1)</u> $(V, (\cdot,\cdot))$ is a Hilbert space $\iff (V, \|\cdot\|)$ is complete.

<u>Examples</u> R^n, $L^2(\Omega)$, $W_2^k(\Omega)$

<u>Def (2.2.3)</u> $S \subset H$ is a subspace of $H \iff$ (1) S is linear

(2) S is closed under $\|\cdot\|$.

<u>Proposition (2.2.4)</u> S is a subspace of $H \Longrightarrow (S, (\cdot,\cdot))$ is a Hilbert space.

<u>Examples of Subspaces</u>

(1) H, $\{0\}$

(2) Let $T: H \to K$ be a cont. linear map, H — H-space, K — linear space

$\ker T = \{v \in H \mid Tv = 0\}$ (see 2.x.1)

(3) Let $x \in H$,

<u>orthogonal complement</u> $x^\perp = \{v \in H \mid (v,x) = 0\}$ is a subspace. $= \operatorname{Ker} L_x$ where $L_x(v) = (v,x)$ linear, bounded.

Proof (i) proof by definition.

(ii) Let $L_x : v \to (v, x)$: $L_x(v) = (v, x)$

$\Longrightarrow$ L_x is linear and bounded $\left(|L_x(v)| \leq \|x\| \, \|v\| \right)$

$x^\perp = \operatorname{Ker} L_x \Longrightarrow x^\perp$ is a subspace.

(4) $M \subset H$ is a subset

orthogonal complement $\quad M^\perp = \{ v \in H \mid (x, v) = 0 \ \forall x \in M \}$ is a subspace.

$$= \bigcap_{x \in M} x^\perp$$

Proposition (2.2.7) $\quad H$ — H-space

2.x.3 (1) $\forall$ subsets $M, N \subset H$, $\quad M \subset N \Longrightarrow N^\perp \subset M^\perp$

(2) $\forall$ subset $M \subset H$ containing $0 \Longrightarrow M \cap M^\perp = \{0\}$

2.x.3 (3) $\{0\}^\perp = H \qquad\qquad M \cap M^\perp \subset \{0\}$

(4) $H^\perp = \{0\}$

Proof (2) $0 \in M$ and, obviously, $0 \in M^\perp \Rightarrow 0 \in M \cap M^\perp$.

$$\forall x \in M \cap M^\perp \Rightarrow \left. \begin{cases} x \in M \Rightarrow M^\perp \subset x^\perp \\ x \in M^\perp \end{cases} \right\} \Rightarrow x \in x^\perp \Rightarrow (x, x) = 0 \Rightarrow x = 0$$

(4) $H^\perp \subset H \Longrightarrow H^\perp = H \cap H^\perp = \{0\}$.

Thm (2.2.8) (Parallelogram Law) $\|\cdot\| = \sqrt{(\cdot,\cdot)}$

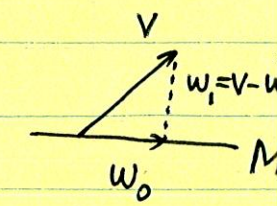
$$\Longrightarrow \|v+w\|^2 + \|v-w\|^2 = 2\left(\|v\|^2 + \|w\|^2\right).$$

(see 2.x.4)

§2.3 Projections onto Subspaces

Propersition (2.3.1) $M \subset H$ – a subspace

$$\forall v \in H,\ \exists w_0 \in H \text{ s.t. } \begin{cases} (1)\ \|v-w_0\| = \inf\limits_{w\in M} \|v-w\| \\ (2)\ w_1 = v - w_0 \in M^{\perp} \end{cases}$$



Proof (1) Let $\delta = \inf\limits_{w\in M} \|v-w\|$

$$\xrightarrow{\text{def. of inf}} \forall n \in I,\ \exists w_n \in M, \text{ s.t. } \begin{cases} \delta \le \|v-w_n\| \\ \|v-w_n\| \le \delta + \frac{1}{n} \end{cases}$$

$$\Longrightarrow \lim_{n\to\infty} \|v-w_n\| = \delta$$

$\{w_n\}$ is a C-seq.

$$\begin{aligned} 0 \le \|w_n - w_m\|^2 &= \|(w_n - v) - (w_m - v)\|^2 \\ &\overset{\text{Parallelogram Law}}{=} 2\left(\|w_n - v\|^2 + \|w_m - v\|^2\right) - \|w_n + w_m - 2v\|^2 \\ & \qquad\qquad - 4\|\underbrace{\tfrac{1}{2}(w_n+w_m)}_{\in M} - v\|^2 \le -4\delta^2 \\ &\le 2\left(\left(\delta + \tfrac{1}{n}\right)^2 + \left(\delta + \tfrac{1}{m}\right)^2\right) - 4\delta^2 \\ &\to 2(\delta^2 + \delta^2) - 4\delta^2 = 0 \end{aligned}$$

$$\underbrace{\|\tfrac{1}{2}(w_n+w_m)}_{\in M} - v\| \ge \inf_{w\in M} \|v-w\| = \delta$$

$$\Longrightarrow w_n \to w_0 \in \overline{M} = M$$

$$\Longrightarrow \|v - w_0\| = \lim \|v - w_n\| = \delta.$$

(2) $\forall\, w \in M$, want to prove $(v-w_0, w) = 0$

$\Rightarrow w_0 + tw \in M \quad \forall\, t \in \mathbb{R}$

$g(0) = \|v-w_0\|^2 \le \|v-(w_0+tw)\|^2 = g(t) \quad \forall\, t \in \mathbb{R}$

$\Rightarrow g(t) = \|v-(w_0+tw)\|^2$ has a min. at $t=0$ $\quad \left(g(0) = \|v-w_0\|^2 \le g(t) \right.$

$\Rightarrow 0 = g'(t)\Big|_{t=0} = -2(v-w_0, w)$ ~~two~~

$\Rightarrow (v-w_0, w) = 0 \quad \forall\, w \in M.$ #

Variational problem Given $v \in H$, Find $w_0 \in M$ s.t.

$$(v-w_0, w) = 0 \;\forall\, w \in M \iff (w_0, w) = (v, w) \;\forall\, w \in M.$$

Minimizational problem Given $v \in H$, Find $w_0 \in M$ s.t.

$$\|v-w_0\| = \inf_{w \in M} \|v-w\|$$

Remark

(M) $\Rightarrow$ (V) : see (2) in Proposition (2.3.1)

(V) $\Rightarrow$ (M) : $\forall\, w \in M$, $\|v-w\|^2 = \|(v-w_0)+(w_0-w)\|^2$

$$= \|v-w_0\|^2 + \|w_0-w\|^2$$

$$\ge \|v-w_0\|^2.$$

Decomposition $\forall\, v \in H$, ~~$\exists\, w_0 \in M$, s.t.~~ the decomposition

$v = w_0 + w_1$ ~~where $w_0 \in M$ and $w_1 = v - w_0 \in M^\perp$~~

is unique, where $w_0 \in M$ and $w_1 = v - w_0 \in M^\perp$.

Proof if $v = w_0 + w_1 = z_0 + z_1 \Rightarrow w_0 - z_0 = z_1 - w_1 \in M \cap M^\perp = \{0\}$

$\Rightarrow w_0 = z_0$ and $w_1 = z_1$.

Projection Operators

$$P_M : H \to M \text{ by } P_M v = \begin{cases} v & \text{if } v \in M \\ w_0 & \text{if } v \in H \setminus M \end{cases} \iff (P_M v, w) = (v, w)\ \forall w$$

$$\cancel{(P_M)^{\perp} : H \to M^{\perp} \text{ by } (P_M)^{\perp} v = \begin{cases} 0 & \text{if } v \in M \\ v - w_0 & \text{if } v \in H \setminus M \end{cases} \iff}$$

$$P_{M^\perp} : H \to M^{\perp} \text{ by } (P_{M^\perp} v, z) = (v, z)\ \forall z \in M^{\perp}$$

$$\Rightarrow P_{M^\perp} v = \begin{cases} 0 & \text{if } v \in M \Leftarrow (P_{M^\perp} v, z) = 0\ \forall z \in M^{\perp} \\ v - w_0 & \text{if } v \in H \setminus M \Leftarrow (P_{M^\perp} v, z) = (v, z) = (v - w_0, z)\ \forall z \end{cases}$$

and $v - w_0 \in M^{\perp}$

<u>Propersition (2.3.5)</u> $M \subset H$ — subspace, $v \in H$

$\exists 1$ decomposition $v = P_M v + P_{M^\perp} v \iff H = M \oplus M^{\perp}$.

<u>Remark</u> (1) P_M and $P_{M^\perp}$ are linear operators

(2) P_M and $P_{M^\perp}$ are projections: $P^2 = P$.

§2.4 Riesz Representation Thrm

$(H, (\cdot,\cdot))$ — Hilbert space

Given $u \in H$, $L_u(v) = (u, v)$ defines a $L_u \in H'$.

Thrm (2.4.2) (R–representation Thrm)

Let H be a H-space, ~~then~~

$\Rightarrow$ (1) $\forall L \in H'$, $\exists 1\, u \in H$ s.t. $L(v) = (u, v)$ $\forall v \in H$.

(2) $\|L\|_{H'} = \|u\|_H$.

Proof (1) Uniqueness $\exists u_1, u_2 \in H$ s.t. $L(v) = (u_i, v)$ $\forall v \in H$

$\Rightarrow 0 = L(v) - L(v) = (u_1 - u_2, v)$ $\forall v \in H$

choosing $v = u_1 - u_2$, $0 = (u_1 - u_2, u_1 - u_2) \Longrightarrow u_1 = u_2$.

Existence Let $M = \ker(L) \Longrightarrow H = M \oplus M^\perp$

$\boxed{u \notin M}$ otherwise $0 = L(u) = (u,u)$ $\Rightarrow u = 0$ $\Rightarrow L(v) = (0, v) =$

Case 1 $M^\perp = \{0\} \Longrightarrow M = H \Longrightarrow L(v) = 0$ $\forall v \in M = H \Longrightarrow L \equiv 0$

$\Longrightarrow$ taking $u = 0 \Longrightarrow L(v) = (0, v) = 0$.

Case 2 $M^\perp \neq \{0\} \Rightarrow \exists z \in M^\perp$, ~~and~~ $z \neq 0$, and $L(z) \neq 0$.

$\boxed{\cancel{M^\perp \text{ is 1-dim}}}$

choose $u = \dfrac{L(z)}{\|z\|_H^2} z \in M^\perp$, then $\forall v \in H$, $v = (v - \beta z) + \beta z \in M \oplus M^\perp$ for $\beta = \dfrac{L(v)}{L(z)}$ $\Longrightarrow L(v - \beta z) = 0$

$(u, v) = (u, (v - \beta z) + \beta z)$ with $\beta = \dfrac{L(v)}{L(z)}$ ~~for any $\beta \in \mathbb{R}$~~

$= (u, \beta z)$ since $u \in M^\perp$, $v - \beta z \in M$

$$= \beta \frac{L(z)}{\|z\|_H^2}(z,z) = \beta L(z) = L(v)$$

$$\Longrightarrow \beta = \frac{L(v)}{L(z)}$$

(2) $u = \frac{L(z)}{\|z\|_H^2} z \Longrightarrow \|u\|_H = \frac{|L(z)|}{\|z\|_H}$

$$\|L\|_{H'} = \sup_{v \in H} \frac{|L(v)|}{\|v\|_H} = |(u,v)| \le \|u\|_H \|v\|_H$$

$$\le \|u\|_H = \frac{|L(z)|}{\|z\|_H} \le \|L\|_{H'}$$

$$\Longrightarrow \|L\|_{H'} = \|u\|_H .$$

#

<u>Remark</u> $\tau : L_u \in H' \longrightarrow u \in H$ is an isometry. $\Longrightarrow H' \cong H$

two different spaces, but $W_2^{-m}(\Omega) \cong W_2^{m}(\Omega)$.

§2.5 Formulation of Symmetric Variational Problems

<u>Def. (2.5.2)</u> (Continuity and Coercivity) $a(\cdot,\cdot) : H \times H \to R$ — bilinear; normed linear spa[ce]

(1) a is bounded $\iff \exists\, C < \infty$ s.t. $|a(u,v)| \le C \|u\|_H \|v\|_H \quad \forall u,v \in H$

(2) a is coercive on $V \subset H \iff \exists\, \alpha > 0$, s.t. $a(v,v) \ge \alpha \|v\|_H^2 \quad \forall v \in H$

~~Proposition (2.5.3) H — a Hilbert space, $a(\cdot,\cdot)$ is cont. on H and $V \subsetneq H$ a subspace~~

Propersition (2.5.2) Let H be a H-space, and $V \subset H$ be a subspace.

Assume that (1) $a(\cdot,\cdot)$ is sym, bilinear,

(2) $a(\cdot,\cdot)$ is cont. on H

(3) $a(\cdot,\cdot)$ is coercive on V

$\Rightarrow \big(V, a(\cdot,\cdot)\big)$ is a H-space.

Proof (1) $a(\cdot,\cdot)$ is an inner product on V.

$\boxed{\|\cdot\|_E \sim \|\cdot\|_H}$

$\left[\alpha \|u\|_H^2 \le a(u,u) = \|u\|_E^2 \le C \|u\|_H^2\right]$

(2) $\left(V, \|\cdot\|_E = a(\cdot,\cdot)^{\frac{1}{2}}\right)$ is complete.

Let $\{v_n\}$ be a C-seq in $(V, \|\cdot\|_E)$

$\|v_n - v_m\|_H \le C\|v_n - v_m\|_E$

$\Rightarrow \{v_n\}$ ———————— $(H, \|\cdot\|_H)$

$\Rightarrow v_n \to v$ in $\|\cdot\|_H$ and $v \in H$ (H is complete)

$\Rightarrow v \in V$ (V is closed in H)

$\|v - v_n\|_E \le C\|v - v_n\|_H \to 0$ $\Rightarrow$ ~~$\lim_{n\to\infty} \|v - v_n\|_E$~~.

$\Rightarrow$ ~~$\{v_n\}$~~ $v_n \to v$ in $\|\cdot\|_E$ and $v \in V \Rightarrow (V, \|\cdot\|_E)$ is complete.

Assumptions (1) $(H, (\cdot,\cdot))$ is a H-space

(2) V is a (closed) subspace of H

(3) $a(\cdot,\cdot)$ is a bounded, sym bilinear form that is coercive on V

Sym Variational Prob. Given $F \in V'$, find $u \in V$ s.t.

$$a(u,v) = F(v) \quad \forall v \in V. \qquad (2.5.5)$$

Example $H = H^1(0,1)$, $V = \{ v \in H^1(0,1) \mid v(0) = 0 \}$

$$a(u,v) = \int_0^1 (uv + u'v')\,dx, \quad F(v) = \int_0^1 f v\,dx$$

Thrm (2.5.6) Under the assumptions (1)–(3), (2.5.5) has a unique solution.

Proof

- $(V, a(\cdot,\cdot))$ is a H-space $\sim (V, \|\cdot\|_H)'$ $\quad |F(v)| \le C\|v\|_H \le C\|v\|_E$
- $\|\cdot\|_E \sim \|\cdot\|_H \Longrightarrow$ "$F \in V' \Rightarrow F \in (V, a(\cdot,\cdot))'$"
- Riesz-R Thrm $\Rightarrow \exists 1\, u \in V$ s.t. $F(v) = a(u,v)$

#

<u>Ritz-Galerkin Approx.</u> Given $V_h \subset V$ and $F \in V'$ where $\dim(V_h) < +\infty$,
Find $u_h \in V_h$ s.t.

$$a(u_h, v) = F(v) \quad \forall v \in V_h \qquad (2.5.7)$$

<u>Thrm</u> (2.5.8) Under the assumptions (1)–(3), (2.5.7) has a unique solution.

<u>Proof</u> $(V_h, a(\cdot,\cdot))$ is a H-space and $F|_{V_h} \in V_h'$. #

<u>Fundamental Orthogonality</u> $a(u-u_h, v) = 0 \quad \forall v \in V_h$.

<u>Corollary</u> $\|u-u_h\|_E = \min\limits_{v\in V_h} \|u-v\|_E$.

<u>Proof</u> $\|u-u_h\|_E^2 = a(u-u_h, u-u_h) = a(u-u_h, u-v) \quad \forall v \in V_h$

$$\leq \|u-u_h\|_E \, \|u-v\|_E$$

$$\Rightarrow \quad \|u-u_h\|_E \leq \inf_{v\in V_h} \|u-v\|_E \leq \|u-u_h\|_E$$ #

Let $Q(v; F) = \frac{1}{2} a(v,v) - F(v)$

<u>Minimization Prob</u> Given $F \in V'$, find $u \in V$ s.t. $Q(u; F) = \min\limits_{v\in V} Q(v; F)$

$\Updownarrow$

$Q(u;F) \leq Q(v;F) \; \forall v \in$

<u>Ritz method</u> Find $u_h \in V_h$, s.t. $Q(u_h; F) = \min\limits_{v\in V_h} Q(v; F)$

<u>Proposition</u> (S) and (M) have the same solution.

<u>Proof</u> "(S)⇒(M)" $\forall v \in V$, $Q(v;F) = Q(u+(v-u); F) \overset{a(u,v-u)-F(v-u)=0}{=} Q(u;F) + \frac{1}{2} a(u-v, u-v) \geq Q(u;F)$

"(M)⇒(S)" $\forall v \in V$, $Q(u;F) \leq Q(u+tv; F) = g(t)$; $\Rightarrow g(t)$ has a min at $t=0 \Rightarrow g'(0) = 0 \Rightarrow a(u,v) - F(v) =$

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§2.6 Formulation of Nonsym. Variational Problems

Assumptions (1) $(H, (\cdot,\cdot))$ is a H-space

(2) V is a closed subspace of H

(3) $a(\cdot,\cdot)$ is a bilinear form on V, but not ~~necessarily~~ sym

(4) $a(\cdot,\cdot)$ is continuous and ~~too~~ coercive on V.

NVP Given $F \in V'$, find $u \in V$ s.t.

$$a(u,v) = F(v) \quad \forall v \in V.$$

Galerkin Approx. ~~G~~iven $V_h \subset V$ and $F \in V'$ with $\dim(V_h) < +\infty$, find $u_h \in V_h$ s.t.

$$a(u_h, v) = F(v) \quad \forall v \in V_h.$$

Example
$$\begin{cases} -u'' + u' + u = f \text{ in } (0,1) \\ u'(0) = u'(1) = 0 \end{cases} \Longrightarrow \begin{cases} V = H^1(0,1) \\ a(u,v) = \int_0^1 (u'v' + u'v + uv)\,dx \\ F(v) = \int_0^1 fv\,dx \end{cases}$$

(1) $a(\cdot,\cdot)$ is cont.

(2) $a(\cdot,\cdot)$ is coercive : $a(v,v) = \frac{1}{2}\int_0^1 (v'+v)^2 dx + \frac{1}{2}\int_0^1 (v'^2 + v^2) dv \geq \frac{1}{2} \|v\|_{H^1}^2$

$$\begin{cases} -u'' + ku' + u = f \text{ in } (0,1) \\ u'(0) = u'(1) = 0 \end{cases}$$ the corresponding $a(\cdot,\cdot)$ is not coercive for large k.

§2.7 The Lax-Milgram Thrm

Lemma (2.7.2) (Contraction Mapping Principle) Let V be a Banach space. Let

nonlinear $T: V \to V$ be a map satisfying

$$\exists M \in [0,1) \text{ s.t. } \|T(v_1) - T(v_2)\| \leq M \|v_1 - v_2\| \quad \forall v_1, v_2 \in V.$$

$$\Longrightarrow \exists 1 \; u \in V \text{ s.t. } u = T(u)$$

(The contraction mapping T has a unique fixed pt u.)

Remark Only need to assume that V is a complete metric space.

Proof Uniqueness $T(u_i) = u_i$, $i = 1, 2$ Assume that $u_1 - u_2 \neq 0$

$$\|u_1 - u_2\| = \|T(u_1) - T(u_2)\| \leq M \|u_1 - u_2\| \; \cancel{\boxed{< \|u_1 - u_2\|}} \Rightarrow 1 \leq M \longrightarrow \text{contradiction}$$

Existence $\begin{cases} \text{Pick } u_0 \in V \\ \text{compute } u_{k+1} = T u_k \end{cases}$

(1) $\{u_k\}$ is a C-seq.

$$\forall N > n, \quad \|u_N - u_n\| = \left\| \sum_{k=n}^{N-1} (u_{k+1} - u_k) \right\| \leq \sum_{k=n}^{N-1} \|u_{k+1} - u_k\|$$

$$\leq \sum_{k=n}^{N-1} M^{k-1} \|u_1 - u_0\| \qquad \left(\begin{aligned} &\|u_{k+1} - u_k\| = \|Tu_k - Tu_{k-1}\| \\ &\leq M \|u_k - u_{k-1}\| \leq \cdots \end{aligned} \right)$$

$$\leq \frac{M^n}{1-M} \|u_1 - u_0\| = \frac{M^n}{1-M} \|Tu_0 - u_0\|$$

(2) V is complete $\Longrightarrow \lim_{n\to\infty} u_n = u \in V$

$$\Longrightarrow u = \lim_{n\to\infty} u_{n+1} = \lim_{n\to\infty} T u_n = T(\lim u_n) = Tu \quad (T \text{ is cont.}) \qquad \#$$

Lax-Milgram Thrm Let $(V, (\cdot,\cdot))$ be a H-space, $a(\cdot,\cdot)$ be a cont., coercive bilinear form, and $F \in V'$,

$$\Rightarrow \exists 1\ u \in V \text{ s.t. } a(u,v) = F(v) \quad \forall v \in V.$$

Proof • $\forall u \in V$, defines a functional $A_u : V \to \mathbb{R}$ by $A_u(v) = a(u,v)$

prove that $A_u \in V'$ (linear, cont) with $\|A_u\|_{V'} \le C\|u\| < +\infty$.

• Define an operator $A : V \to V'$ by $Au = A_u$

A is linear: $A(\alpha u_1 + \beta u_2)(v) = a(\alpha u_1 + \beta u_2, v) = (\alpha A u_1 + \beta A u_2)(v)$

A is cont. $\|A\|_{L(V,V')} = \sup_{u \in V} \dfrac{\|Au\|_{V'}}{\|u\|_V} = \|A_u\|_{V'} \le C$.

$$\Longleftrightarrow \exists 1\ u \in V, \text{ s.t. } Au(v) = F(v) \quad \forall v \in V$$

$$\Updownarrow$$

$\exists 1\ u \in V$ s.t. $\underline{Au = F}$ in V'

RR Thrm $\Longrightarrow$ $\Updownarrow$

$\exists 1\ u \in V$ s.t. $\tau \underline{Au = \tau F}$ in V

$\phi(v) = (\tau\phi, v)$ — $\|\tau\phi\|_V = \|\phi\|$

τ is linear

$(\tau : V' \to V$ isometric mappin

• Find $\rho \ne 0$ s.t. $Tv \equiv v - \rho(\tau A v - \tau F) : V \to V'$ is contractive

$$\left[\Longrightarrow \exists 1\ u \in V \text{ s.t. } Tu = u \Longleftrightarrow \tau A u = \tau F \right]$$

$\forall v_1, v_2 \in V$, let $v = v_1 - v_2$

$\|Tv_1 - Tv_2\|^2 = \|v - \rho\tau A v\|^2$ (τ, A are linear)

$= \|v\|^2 - 2\rho(\tau A v, v) + \rho^2 \|\tau A v\|^2$ — $(\tau A v, v) = Av(v) = a(v,v)$

$= \|v\|^2 - 2\rho\, a(v,v) + \rho^2 a(v, \tau A v)$ — $(\tau A v, \tau A v) = Av(\tau A v) = a(v, \tau A v)$

$$\leq \|v\|^2 - 2\rho\alpha\|v\|^2 + \rho^2 \|v\|\,\|\tau A v\| \qquad \left(\|\tau A v\|_V = \|Av\|_{V'} \leq C\|v\| \right.$$

$$\leq \left(1 - 2\rho\alpha + C\rho^2\right)\|v\|^2$$

$$\equiv M^2\|v\|^2 \qquad \Longrightarrow \qquad \|Tv\| \leq M\|v\| .$$

Want $M^2 = 1 - 2\rho\alpha + C\rho^2 < 1$ for some ρ

$$\Leftrightarrow \rho(\rho C - 2\alpha) < 0 \xRightarrow{\begin{cases}\rho>0\\ \rho C-2\alpha<0\end{cases}} \rho \in \left(0, \frac{2\alpha}{C}\right). \qquad \#$$

Corollary
- (0) Regularity estimate: $\|u\|_V \leq \frac{1}{\alpha}\|F\|_{V'}$ $\quad \alpha\|u\|_V^2 \leq a(u,u) = F(u)$
- (1) (NVP) has a unique solution $u \in V$.
- (2) (NVP_h) ———— $u_h \in V_h$.
- (3) V_h need not be finite-dimensional.

§2.8 Estimates for General FEA

NVP Find $u \in V$ s.t. $a(u,v) = F(v)\ \forall v \in V$

NVP_h Find $u_h \in V_h \subset V$ s.t. $a(u_h, v) = F(v)\ \forall v \in V$.

$$\Longrightarrow a(u - u_h, v) = 0\ \forall v \in V_h \quad \text{\underline{the error equation}}$$

Thrm (2.8.1) (Céa's Lemma) $\|u - u_h\|_V \leq \frac{C}{\alpha} \cancel{\inf} \min_{v \in V_h} \|u - v\|_V$.

Proof $\alpha\|u - u_h\|_V^2 \leq a(u - u_h, u - u_h) = a(u - u_h, u - v) \quad \forall v \in V_h$

$$\leq C\|u - u_h\|_V \|u - v\|_V \qquad \#$$