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Nonconforming quadrilateral finite elements: a correction

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Abstract. The object of this note is to correct an error in defining the basis for some nonconforming finite element methods when quadrilaterals occur in the partition of a two-dimensional domain.

1 Introduction

In two recent papers [2, 3], the authors described a basis over a quadrilateral for nonconforming finite element methods for a second order elliptic problem [3] and for stationary Stokes and Navier–Stokes problems [2]. The methods given therein are stable, but a loss of accuracy will occur if a true quadrilateral (i.e., one having two opposite, nonparallel edges) is included in the partition. There is no loss associated with rectangles or parallelograms. The loss caused by nonparallel opposite edges was first noted by Rannacher and Turek [4] for a basis different from ours; very recently, Arnold, Boffi, and Falk [1] showed that, in order to have second order accuracy on a true quadrilateral, it is necessary that the basis on the reference square contain the full bilinear basis. This was not the case in our papers, and the simplest correction for this error is provided in the next section.

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2 The corrected basis

Let the reference element be the square $\widehat{R} = [-1, 1]^2$, and let

$$\varphi_\ell(x) = \begin{cases} x^2 - \frac{5}{3}x^4, & \ell = 1, \\ x^2 - \frac{25}{6}x^4 + \frac{7}{2}x^6, & \ell = 2. \end{cases} \tag{2.1}$$

The two reference bases in [2,3] were defined by

$$\mathcal{Q}_\ell^*(\widehat{R}) = \mathrm{Span}\,\{1, x, y, \varphi_\ell(x) - \varphi_\ell(y)\}, \quad \ell = 1, 2. \tag{2.2}$$

Clearly, the xy-term in the bilinear basis is missing. The correction is to add this function to the basis whenever the element in the partition is a true quadrilateral; i.e., let

$$\mathcal{Q}_\ell(\widehat{R}) = \mathrm{Span}\,\{1, x, y, xy, \varphi_\ell(x) - \varphi_\ell(y)\}, \quad \ell = 1, 2. \tag{2.3}$$

The degrees of freedom associated with $\mathcal{Q}_\ell(\widehat{R})$ are given by the values of a function f at the four midpoints of the edges of $\widehat{R}$ and the integral

$$\int_{\widehat{R}} f(x, y) xy \, dx dy.$$

Then, if an element $E = F(\widehat{R})$, where F is bilinear, is a true quadrilateral, let

$$\mathcal{Q}_\ell(E) = \{v \,:\, v = \hat{v} \circ F^{-1},\ \hat{v} \in \mathcal{Q}_\ell(\widehat{R})\}, \quad \ell = 1, 2. \tag{2.4}$$

For all other elements E, let

$$\mathcal{Q}_\ell(E) = \{v \,:\, v = \hat{v} \circ F^{-1},\ \hat{v} \in \mathcal{Q}_\ell^*(\widehat{R})\}, \quad \ell = 1, 2. \tag{2.5}$$

In both of our papers, the nonconforming Galerkin space was obtained by requiring continuity at the midpoints of the edges of the elements; the basis elements corresponding to the xy-term vanish at these points; thus, the added element is a bubble and can be eliminated trivially before solving the the algebraic equations related to the other entries.

With this modification in the methods proposed in [2,3], all convergence results given therein become valid.

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