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LEAST SQUARES FOR THE PERTURBED STOKES EQUATIONS AND THE REISSNER–MINDLIN PLATE*

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Abstract. In this paper, we develop two least-squares approaches for the solution of the Stokes equations perturbed by a Laplacian term. (Such perturbed Stokes equations arise from finite element approximations of the Reissner–Mindlin plate.) Both are two-stage algorithms that solve first for the curls of the rotation of the fibers and the solenoidal part of the shear strain, then for the rotation itself (if desired). One approach uses L^2 norms and the other approach uses H^{-1} norms to define the least-squares functionals. It is shown that the H^{-1} norm approach, under general assumptions, and the L^2 norm approach, under certain H^2 regularity assumptions, admit optimal performance for standard finite element discretization and either standard multigrid solution methods or preconditioners. These methods do not degrade when the perturbed parameter (the plate thickness) approaches zero. We also develop a three-stage least-squares method for the Reissner–Mindlin plate, which first solves for the curls of the rotation and the shear strain, next for the rotation itself, and then for the transverse displacement.

Key words. least squares, Reissner–Mindlin model, perturbed Stokes equations, finite element methods, multigrid methods

AMS subject classifications. 65F10, 65F30

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1. Introduction.

Let Ω be the region occupied by the midsection of the plate and assume that Ω is a bounded, open, connected domain in $\mathcal{R}^2$ with Lipschitz boundary $\partial\Omega$. Let ω and $\boldsymbol{\phi} = (\phi_1, \phi_2)^t$ denote the transverse displacement of Ω and the rotation of the fibers normal to Ω, respectively. The strong form of the hard clamped Reissner–Mindlin plate model is given by

$$(1.1)\quad \begin{cases} -\frac{E}{24(1+\nu)}\Delta\boldsymbol{\phi} - \frac{E}{24(1-\nu)}\nabla(\nabla\cdot\boldsymbol{\phi}) + \lambda t^{-2}(\boldsymbol{\phi} - \nabla\omega) &= \mathbf{0} \quad \text{in} \quad \Omega, \\ \lambda t^{-2}\nabla\cdot(\boldsymbol{\phi} - \nabla\omega) &= g \quad \text{in} \quad \Omega \end{cases}$$

with boundary conditions

$$(1.2)\qquad \boldsymbol{\phi} = \mathbf{0} \quad \text{and} \quad \omega = 0 \quad \text{on} \quad \partial\Omega,$$

where the symbols Δ, ∇, and $\nabla\cdot$ stand for the Laplacian, gradient, and divergence operators, respectively ($\Delta\,\boldsymbol{\phi}$ signifies the 2-vector of components $\Delta\,\phi_i$; that is, Δ applies to $\boldsymbol{\phi}$ componentwise); $t > 0$ is the *plate thickness*; $\lambda = Ek/2(1+\nu)$ is the shear modulus with E the Young's modulus, $\nu \in (-1, \frac{1}{2})$ the Poisson ratio, and k the shear correction factor; and g is the given scaled transverse loading function (scaling by a constant multiple of the square of the thickness [6]). Without loss of generality, assume that $\lambda = 1$.

It is known that standard finite element approximations grossly underestimate the displacement of very thin plates. Such a phenomenon is referred to as *locking* of the numerical solution. There have been many studies to develop alternate approaches

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(see [1], [6], [7], [8], [9]) that are robust in the zero limit of the plate thickness t. Among those *locking free* discretization schemes, the fundamental work of Brezzi and Fortin in [6] must be noted. They introduced a *three-stage* finite element method based on the following Helmholtz decomposition of the *transverse shear strain* vector:

$$t^{-2}(\nabla\omega - \boldsymbol{\phi}) = \nabla r - \nabla^{\perp} p,$$

where

$$\nabla^{\perp} p = \begin{pmatrix} \partial_2 p \\ -\partial_1 p \end{pmatrix}.$$

The $\nabla^{\perp}$ is the formal adjoint operator of the standard two-dimensional curl operator, $\nabla\times$, which means that the curl of $\boldsymbol{\phi}$ is the scalar function

$$\nabla\times\boldsymbol{\phi} = \partial_1\phi_2 - \partial_2\phi_1.$$

The first and third stages solve simple Poisson equations for the respective r and ω. The second stage uses mixed finite element methods to solve a Stokes equation perturbed by a Laplacian term for the $\boldsymbol{\phi}$ and p:

$$\text{(1.3)} \qquad \begin{cases} -\alpha_1\Delta\boldsymbol{\phi} - \alpha_2\nabla(\nabla\cdot\boldsymbol{\phi}) + \nabla^{\perp} p &= \mathbf{f} \quad \text{in} \quad \Omega, \\ \nabla\times\boldsymbol{\phi} + t^2\Delta p &= tf_3 + f_4 \quad \text{in} \quad \Omega \end{cases}$$

with boundary conditions

$$\text{(1.4)} \qquad \boldsymbol{\phi} = \mathbf{0} \quad \text{and} \quad \nabla p\cdot\mathbf{n} = 0 \quad \text{on} \quad \partial\Omega.$$

Here

$$\alpha_1 = \frac{E}{24(1+\nu)} > 0, \quad \alpha_2 = \frac{E}{24(1-\nu)} > 0, \quad \mathbf{f} = \nabla r, \quad \text{and} \quad f_3 = f_4 = 0,$$

and $\mathbf{n} = (n_1, n_2)^t$ is the outward unit vector normal to the boundary $\partial\Omega$. Such a three-stage algorithm converges uniformly in the thickness t. (For $t = 0$, (1.3) is the usual Stokes equations subject to the change of variables $(\hat{x}_1, \hat{x}_2) = (x_2, -x_1)$ and the transformation $(\hat{\phi}_1, \hat{\phi}_2)^t = (\phi_2, -\phi_1)^t$.) As usual, mixed finite element methods are subject to the inf-sup condition and the resulting algebraic equations are difficult to solve. Indeed, little attention seems to have been paid to the development of robust solution strategies for the resulting algebraic equations. (See [2] for the recent treatment of preconditioners for the resulting discrete problem based on the discrete Helmholtz decomposition.)

Recently, there has been substantial interest in the use of least-squares principles for numerical approximations of elliptic partial differential equations and systems. See [12], [13], [14] for linear elasticity equations which are parameter dependent problems. Its advantages over the usual mixed finite element discretizations include that the choice of finite element spaces is not subject to the inf-sup condition, that the resulting algebraic equations can be efficiently solved by standard multigrid methods or preconditioned by well-known techniques, and that the value of the least-squares functional provides a good error indicator which can be used efficiently in a refinement process. In [15], we proposed and analyzed least-squares methods for approximating the primitive variables, $\boldsymbol{\phi}$ and p, of the perturbed Stokes equations (1.3)–(1.4). It was shown that optimal discretization error estimates are uniform in t and that the resulting algebraic equations can be uniformly well preconditioned by well-known techniques in the thickness.

What is often needed in practice is the stress. Those variables can be obtained by differentiating $\boldsymbol{\phi}$, but this weakens the *order* and *strength* of the approximation. The purpose of this paper is to develop two least-squares approaches directly approximating the *rotation flux*, $\nabla^{\perp}\boldsymbol{\phi}^t$, for the perturbed Stokes equations. One approach uses L^2 norms to define the least-squares functional and it yields uniform and optimal H^1 approximations of all variables under certain H^2 regularity assumptions. The other approach is based on the least-squares functional involving H^{-1} norms and yields uniform and optimal L^2 approximations for the rotation flux and H^1 approximations for p. Note that least-squares functionals proposed in this paper are similar to those in [13] for the Stokes equation perturbed by p. They differ in the weight and norm of the residual of the equation involving trace. This is because for the solution of problem (1.3), which is the perturbation of the Stokes equation by the Laplace of p, $\frac{1}{t}\nabla\times\boldsymbol{\phi} = t\Delta p$ and its derivative has different scales. Even though any of approaches for the perturbed Stokes equations combining with the three-stage algorithm introduced in [6] leads to an approach for the Reissner–Mindlin plate, we will also develop a direct approach for the Reissner–Mindlin plate. Such a method solves for the rotation flux and transverse shear strain variables first. The rotation components and the transverse displacement component can then be obtained as solutions of scalar Poisson equations (if desired).

The paper is organized as follows. Section 2 establishes uniform regularity estimates for the perturbed Stokes equations (1.3)–(1.4) in t and section 3 introduces the equivalent first- and second-order systems. We establish ellipticity and continuity of the homogeneous least-squares functionals and its discrete counterparts in sections 4 and 5, respectively. Section 6 studies finite element approximations based on the discrete least-squares functionals and establishes error estimates. Finally, we consider a direct least-squares approach for the Reissner–Mindlin plate in section 7.

We will use the standard notation and definition for the Sobolev spaces $H^s(\Omega)$ for $s \geq 0$; the associated inner products are denoted by $(\cdot,\cdot)_{s,\Omega}$, and their norms by $\|\cdot\|_{s,\Omega}$. (We will omit Ω from the inner product and norm designation when there is no risk of confusion.) For $s=0$, $H^s(\Omega)$ coincides with $L^2(\Omega)$. In this case, the norm and inner product will be denoted by $\|\cdot\|$ and $(\cdot,\cdot)$, respectively. As usual, $H_0^1(\Omega)$ is the closure of $\mathcal{D}(\Omega)$, the linear space of infinitely differentiable functions with compact support on Ω, with respect to the norm $\|\cdot\|_1$. Let $H_0^{-1}(\Omega)$ and $H^{-1}(\Omega)$ be duals of the respective $H_0^1(\Omega)$ and $H^1(\Omega)$ with norms defined by

$$\|\psi\|_{-1,0} = \sup_{0\neq\phi\in H_0^1(\Omega)} \frac{(\psi,\phi)}{\|\phi\|_1} \quad \text{and} \quad \|\psi\|_{-1} = \sup_{0\neq\phi\in H^1(\Omega)} \frac{(\psi,\phi)}{\|\phi\|_1},$$

respectively. Define the product spaces $H_0^1(\Omega)^n = \prod_{i=1}^n H_0^1(\Omega)$ with standard product norms and let $L_0^2(\Omega)$ denote the subspace of $L^2(\Omega)$ consisting of all such functions in $L^2(\Omega)$ having mean value zero.

2. Regularity estimates.

This section establishes regularity estimates for the perturbed Stokes equations in (1.3)–(1.4), for which we will need in the subsequent sections. Note that the forcing function in the second equation of (1.3) has a special form. When $f_3 = f_4 = 0$, the H^2 regularity estimate for (1.3)–(1.4) was derived in [6] (see also [1]). To this end, let us write (1.3)–(1.4) in the variational form, i.e., to find

$(\boldsymbol{\phi}, p) \in H_0^1(\Omega)^2 \times (H^1(\Omega)/\mathcal{R})$ such that

$$\text{(2.1)} \quad \left\{ \begin{array}{rcll} a(\boldsymbol{\phi}, \boldsymbol{\psi}) + (\nabla^\perp p, \boldsymbol{\psi}) & = & (\mathbf{f}, \boldsymbol{\psi}) & \forall\ \boldsymbol{\psi} \in H_0^1(\Omega)^2, \\ (\boldsymbol{\phi}, \nabla^\perp q) - t^2(\nabla^\perp p, \nabla^\perp q) & = & (tf_3, q) + (f_4, q) & \forall\ q \in H^1(\Omega)/\mathcal{R}, \end{array} \right.$$

where the bilinear form $a(\cdot,\cdot)$ is given by

$$a(\boldsymbol{\phi}, \boldsymbol{\psi}) = \alpha_1(\nabla\boldsymbol{\phi}, \nabla\boldsymbol{\psi}) + \alpha_2(\nabla\cdot\boldsymbol{\phi}, \nabla\cdot\boldsymbol{\psi}).$$

Below, we will use C with or without subscripts to denote a generic positive constant, possibly different at different occurrences, which is independent of the plate thickness t and the mesh size h introduced in the subsequent section but may depend on the domain Ω. We will frequently use the term *uniform* in reference to a relation to mean that it holds independent of t and h.

THEOREM 2.1. *For any $0 < t \le C$ and any given functions $\mathbf{f} \in H_0^{-1}(\Omega)^2$, $f_3 \in H^{-1}(\Omega)$, and $f_4 \in L^2(\Omega)$, assume that f_3 and f_4 satisfy the compatibility condition*

$$\int_\Omega (tf_3 + f_4)\, dx = 0.$$

Then there exists a unique solution, $(\boldsymbol{\phi}, p)$, of problem (2.1) *in $H_0^1(\Omega)^2 \times \big(H^1(\Omega)/\mathcal{R}\big)$. Moreover,*

(1) *there exists a positive constant C independent of t, $\mathbf{f}$, f_3, and f_4 such that*

$$\text{(2.2)} \qquad \|\boldsymbol{\phi}\|_1 + \|p\| + \|t\nabla p\| \le C\left(\|\mathbf{f}\|_{-1,0} + \|f_3\|_{-1} + \|f_4\|\right);$$

(2) *if the domain Ω is a convex polygon or has $C^{1,1}$ boundary and if $\mathbf{f} \in L^2(\Omega)^2$, $f_3 \in L^2(\Omega)$, and $f_4 \in H^1(\Omega)$, then $(\boldsymbol{\phi}, p)$ is in $H^2(\Omega)^2 \times \big(H^2(\Omega)/\mathcal{R}\big)$ and there exists a positive constant C independent of t, $\mathbf{f}$, f_3, and f_4 such that*

$$\text{(2.3)} \qquad \|\boldsymbol{\phi}\|_2 + \|p\|_1 + \|t\Delta p\| \le C\left(\|\mathbf{f}\| + \|f_3\| + \|f_4\|_1\right).$$

Proof. Existence and uniqueness immediately follow from the Lax–Milgram theorem. To prove the H^1 regularity in (2.2), choosing $\boldsymbol{\psi} = \boldsymbol{\phi}$ and $q = p$ in (2.1), subtracting the second equation from the first equation in (2.1), and using the Poincaré–Friedrichs and Cauchy–Schwarz inequalities and the definition of H^{-1} norms, we have that

$$\begin{aligned} \|\boldsymbol{\phi}\|_1^2 + \|tp\|_1^2 &\le C\left(a(\boldsymbol{\phi}, \boldsymbol{\phi}) + \|t\nabla^\perp p\|^2\right) \\ &= C\left((\mathbf{f}, \boldsymbol{\phi}) - (f_3, tp) - (f_4, p)\right) \\ &\le C\left(\|\mathbf{f}\|_{-1,0}\|\boldsymbol{\phi}\|_1 + \|f_3\|_{-1}\|tp\|_1 + \|f_4\|\|p\|\right). \qquad \text{(2.4)} \end{aligned}$$

Since $p \in H^1(\Omega)/\mathcal{R}$, note first that by using the well-known inequality (see, e.g., [18]), $\|p\| \le C\|\nabla p\|_{-1,0}$, and the change of variable, $(\hat{x}_1, \hat{x}_2) = (x_2, -x_1)$, we have that

$$\text{(2.5)} \qquad \|p\| = \|\hat{p}\| \le C\|\hat{\nabla}\hat{p}\|_{-1,0} = C\|\nabla^\perp p\|_{-1,0}.$$

It then follows from the first equation in (1.3) and the easily established bounds

$$\|\Delta\boldsymbol{\phi}\|_{-1,0} \le \|\nabla\boldsymbol{\phi}\|_1 \quad \text{and} \quad \|\nabla\nabla\cdot\boldsymbol{\phi}\|_{-1,0} \le \|\nabla\boldsymbol{\phi}\|_1$$

that

$$\text{(2.6)} \qquad \|p\| \le C\left(\|\mathbf{f}\|_{-1,0} + \|\boldsymbol{\phi}\|_1\right).$$

Now by (2.4), (2.6), and the ϵ-inequality, we obtain that

$$\begin{aligned}\|\boldsymbol{\phi}\|_1^2 + \|tp\|_1^2 &\le C\Big((\|\mathbf{f}\|_{-1,0} + \|f_4\|)\|\boldsymbol{\phi}\|_1 + \|f_3\|_{-1}\|tp\|_1 + \|\mathbf{f}\|_{-1,0}\|f_4\|\Big)\\ &\le C\left(\|\mathbf{f}\|_{-1,0}^2 + \|f_3\|_{-1}^2 + \|f_4\|^2\right) + \frac{1}{2}\left(\|\boldsymbol{\phi}\|_1^2 + \|tp\|_1^2\right).\end{aligned}$$

Hence,

$$\|\boldsymbol{\phi}\|_1 + \|tp\|_1 \le C\left(\|\mathbf{f}\|_{-1,0} + \|f_3\|_{-1} + \|f_4\|\right),$$

which, together with (2.6), implies the validity of (2.2).

To prove H^2 regularity (2.3), we use a similar proof as that in [6] (see also [1]). Such proof is based on the H^2 regularity of the standard Stokes equations. To this end, let $(\boldsymbol{\phi}^0, p^0) \in H_0^1(\Omega)^2 \times L_0^2(\Omega)$ be the solution of (2.1) with $t = 0$ and $q \in L_0^2(\Omega)$. Then the well-known H^2 regularity estimate for the Stokes equations (see [20] and [19]) gives that

$$\|\boldsymbol{\phi}^0\|_2 + \|p^0\|_1 \le C\left(\|\mathbf{f}\| + \|f_4\|_1\right). \tag{2.7}$$

Denote $\boldsymbol{\phi}^* = \boldsymbol{\phi} - \boldsymbol{\phi}^0$ and $p^* = p - p^0$, then $(\boldsymbol{\phi}^*, p^*) \in H_0^1(\Omega)^2 \times \left(H^1(\Omega)/\mathcal{R}\right)$ satisfies the following system:

$$\begin{cases} a(\boldsymbol{\phi}^*, \boldsymbol{\psi}) + (\nabla^\perp p^*, \boldsymbol{\psi}) = 0 & \forall\ \boldsymbol{\psi} \in H_0^1(\Omega)^2,\\ (\boldsymbol{\phi}^*, \nabla^\perp q) - t^2(\nabla^\perp p^*, \nabla^\perp q) = (tf_3, q) + t^2(\nabla^\perp p^0, \nabla^\perp q) & \forall\ q \in H^1(\Omega)/\mathcal{R}.\end{cases} \tag{2.8}$$

Since

$$\|p^*\| \le C\|\nabla^\perp p^*\|_{-1,0} \le C\|\boldsymbol{\phi}^*\|_1,$$

the same argument as that in the proof of (2.2) leads to

$$\|\boldsymbol{\phi}^*\|_1^2 + \|tp^*\|_1^2 \le C\left(\|f_3\|\|tp^*\| + \|tp^0\|_1\|tp^*\|_1\right) \le Ct\left(\|f_3\|\|\boldsymbol{\phi}^*\|_1 + \|p^0\|_1\|tp^*\|_1\right),$$

which, together with the Cauchy–Schwarz inequality and (2.7), implies that

$$\|\boldsymbol{\phi}^*\|_1 + \|tp^*\|_1 \le Ct\left(\|f_3\| + \|p^0\|_1\right) \le Ct\left(\|\mathbf{f}\| + \|f_3\| + \|f_4\|_1\right). \tag{2.9}$$

It then follows from the triangle inequality, (2.7), and (2.9) that

$$\|p\|_1 \le \|p^*\|_1 + \|p^0\|_1 \le C\left(\|\mathbf{f}\| + \|f_3\| + \|f_4\|_1\right).$$

Now the first equation in (2.1) gives that

$$\|\boldsymbol{\phi}\|_2 \le C\|\alpha_1\boldsymbol{\Delta\phi} + \alpha_2\nabla\nabla\cdot\boldsymbol{\phi}\| = C\|\mathbf{f} - \nabla^\perp p\| \le C\left(\|\mathbf{f}\| + \|p\|_1\right) \le C\left(\|\mathbf{f}\| + \|f_3\| + \|f_4\|_1\right).$$

Finally, note from the second equation in (2.8) that

$$t^2\Delta p = -\nabla\times\boldsymbol{\phi}^* + tf_3;$$

by using the triangle inequality and (2.9) we then have that

$$\|t\Delta p\| \le \left\|\frac{1}{t}\nabla\times\boldsymbol{\phi}^*\right\| + \|f_3\| \le C\left(\|\mathbf{f}\| + \|f_3\| + \|f_4\|_1\right).$$

This completes the proof of (2.3) and, hence, the theorem.

3. First- and second-order systems. We will be introducing a new independent variable related to the 2^2-vector function of $\nabla^\perp$ of the ϕ_i, $i = 1, 2$. It will be convenient to view the original 2-vector functions as column vectors and the new 2^2-vector functions as block column vectors or matrices. Thus, given

$$\boldsymbol{\phi} = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$$

and denoting $\boldsymbol{\phi}^t = (\phi_1, \phi_2)$, then an operator G defined on scalar functions (e.g., $G = \nabla^\perp$) is extended to 2-vectors componentwise:

$$G\boldsymbol{\phi}^t = (G\phi_1, G\phi_2)$$

and

$$G\boldsymbol{\phi} = \begin{pmatrix} G\phi_1 \\ G\phi_2 \end{pmatrix}.$$

If $\mathbf{U}_i \equiv G\,\phi_i$ is a vector function, then we write the matrix

$$\begin{aligned} \mathbf{U} &\equiv G\boldsymbol{\phi}^t = (\mathbf{U}_1, \mathbf{U}_2) \\ &= \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix}. \end{aligned}$$

We then define the trace operator tr according to

$$\mathrm{tr}\mathbf{U} = U_{11} + U_{22}.$$

If D is an operator (e.g., $\nabla\cdot$) on vector functions, then its extension to matrices is defined by

$$D\mathbf{U} = (D\mathbf{U}_1, D\mathbf{U}_2).$$

For example, writing $\mathbf{U}_i = (U_{i1},\, U_{i2})^t$, then

$$\nabla \times \mathbf{U} = (\partial_1 U_{12} - \partial_2 U_{11}, \partial_1 U_{22} - \partial_2 U_{21}).$$

We also extend the respective normal and tangential operators $\mathbf{n}\cdot$ and $\boldsymbol{\tau}\cdot$ componentwise:

$$\mathbf{n}\cdot\mathbf{U} = (\mathbf{n}\cdot\mathbf{U}_1, \mathbf{n}\cdot\mathbf{U}_2) \quad \text{and} \quad \boldsymbol{\tau}\cdot\mathbf{U} = (\boldsymbol{\tau}\cdot\mathbf{U}_1, \boldsymbol{\tau}\cdot\mathbf{U}_2),$$

where $\boldsymbol{\tau} = (-n_2, n_1)^t$ is the unit counterclockwise vector tangent to the boundary $\partial\Omega$. Finally, inner products and norms on the matrix functions are defined in the natural componentwise way:

$$\|\mathbf{U}\|^2 = \sum_{i=1}^{2} \|\mathbf{U}_i\|^2.$$

We introduce the *rotation flux* variable $\mathbf{U} = \nabla^\perp\boldsymbol{\phi}^t$, that is,

$$\mathbf{U} = (U_{ij})_{2\times 2} = (\nabla^\perp\phi_1, \nabla^\perp\phi_2);$$

the definition of $\mathbf{U}$ and the homogeneous Dirichlet boundary condition of $\boldsymbol{\phi}$ imply that

$$\nabla \cdot \mathbf{U} = \mathbf{0}, \quad tr\mathbf{U} = -\nabla\times\boldsymbol{\phi} \quad \text{in } \Omega \quad \text{and} \quad \mathbf{n}\cdot\mathbf{U} = \mathbf{0}^t \quad \text{on } \partial\Omega. \tag{3.1}$$

Using the identity

$$\nabla(\nabla\cdot\boldsymbol{\phi}) = \Delta\boldsymbol{\phi} + \nabla^{\perp}(\nabla\times\boldsymbol{\phi}), \tag{3.2}$$

the first equation in (1.3) may be rewritten as follows:

$$-\alpha\Delta\boldsymbol{\phi} - \alpha_2\nabla^{\perp}(\nabla\times\boldsymbol{\phi}) + \nabla^{\perp}p = \mathbf{f},$$

where $\alpha = \alpha_1 + \alpha_2$. Hence, a system that is equivalent to (1.3) is

$$\left\{ \begin{array}{rcll} \mathbf{U} - \nabla^{\perp}\boldsymbol{\phi}^t & = & \mathbf{O} & \text{in} \quad \Omega, \\ \alpha(\nabla\times\mathbf{U})^t + \alpha_2\nabla^{\perp}(tr\mathbf{U}) + \nabla^{\perp}p & = & \mathbf{f} & \text{in} \quad \Omega, \\ -tr\mathbf{U} + t^2\Delta p & = & tf_3 + f_4 & \text{in} \quad \Omega, \\ \nabla\cdot\mathbf{U} & = & \mathbf{0}^t & \text{in} \quad \Omega \end{array} \right. \tag{3.3}$$

with boundary conditions

$$\boldsymbol{\phi} = \mathbf{0}, \quad \mathbf{n}\cdot\nabla p = 0, \quad \text{and} \quad \mathbf{n}\cdot\mathbf{U} = \mathbf{0}^t \quad \text{on } \partial\Omega. \tag{3.4}$$

We will show in the next section that this extended system is well posed and suitable for treatment by least-squares principles. However, we will mainly consider the system that involves p and $\mathbf{U}$:

$$\left\{ \begin{array}{rcll} \alpha(\nabla\times\mathbf{U})^t + \alpha_2\nabla^{\perp}(tr\mathbf{U}) + \nabla^{\perp}p & = & \mathbf{f} & \text{in} \quad \Omega, \\ -tr\mathbf{U} + t^2\Delta p & = & tf_3 + f_4 & \text{in} \quad \Omega, \\ \nabla\cdot\mathbf{U} & = & \mathbf{0}^t & \text{in} \quad \Omega \end{array} \right. \tag{3.5}$$

with boundary conditions

$$\mathbf{n}\cdot\nabla p = 0 \quad \text{and} \quad \mathbf{n}\cdot\mathbf{U} = \mathbf{0}^t \quad \text{on } \partial\Omega, \tag{3.6}$$

since it has less independent variables than that of system (3.3)–(3.4). If the rotation $\boldsymbol{\phi}$ is desired, it can then be obtained as solutions of two scalar Poisson equations (see section 6).

Let

$$H(\text{div};\Omega) = \{\mathbf{v} \in L^2(\Omega)^2 : \ \nabla\cdot\mathbf{v} \in L^2(\Omega)\}$$

and

$$H(\text{curl};\Omega) = \{\mathbf{v} \in L^2(\Omega)^2 : \ \nabla\times\mathbf{v} \in L^2(\Omega)\},$$

which are Hilbert spaces under the respective norms:

$$\|\mathbf{v}\|_{H(\text{div};\Omega)} = \left(\|\mathbf{v}\|^2 + \|\nabla\cdot\mathbf{v}\|^2\right)^{\frac{1}{2}} \quad \text{and} \quad \|\mathbf{v}\|_{H(\text{curl};\Omega)} = \left(\|\mathbf{v}\|^2 + \|\nabla\times\mathbf{v}\|^2\right)^{\frac{1}{2}}.$$

Define their respective subspaces:

$$H_0(\text{div};\Omega) = \{\mathbf{v} \in H(\text{div};\Omega) : \ \mathbf{n}\cdot\mathbf{v} = 0 \text{ on } \partial\Omega\}$$

and

$$H_0(\mathrm{curl};\Omega) = \{\mathbf{v} \in H(\mathrm{curl};\Omega) : \ \mathbf{n}\times\mathbf{v} = 0 \text{ on } \partial\Omega\}.$$

Set

$$W = H_0(\mathrm{div};\Omega) \cap H(\mathrm{curl};\Omega).$$

We will also make use of the following results (see [18]).

THEOREM 3.1. *Assume that the domain* Ω *is a bounded convex polygon or has* $C^{1,1}$ *boundary. Then for any vector function* $\mathbf{v}$ *in* W*, we have*

$$\|\mathbf{v}\|_1^2 \le C\left(\|\mathbf{v}\|^2 + \|\nabla\cdot\mathbf{v}\|^2 + \|\nabla\times\mathbf{v}\|^2\right). \tag{3.7}$$

If, in addition, the domain is simply connected, then

$$\|\mathbf{v}\|_1^2 \le C\left(\|\nabla\cdot\mathbf{v}\|^2 + \|\nabla\times\mathbf{v}\|^2\right). \tag{3.8}$$

4. Least-squares functionals. The primary objective of this section is to establish ellipticity and continuity of the homogeneous least-squares functionals based on (3.5) and (3.3) with $f_3 = f_4 = 0$ in appropriate Sobolev spaces. To this end, let

$$\mathcal{V}_{-1} = \mathcal{P}_1 \times H_0(\mathrm{div};\Omega)^2 \quad \text{and} \quad \mathcal{V}_0 = \mathcal{P}_2 \times W \times W,$$

where

$$\mathcal{P}_1 = \{q \in H^1(\Omega) \,:\, \mathbf{n}\cdot\nabla q = 0 \quad \text{on } \partial\Omega\}$$

and

$$\mathcal{P}_2 = \{q \in H^2(\Omega) \,:\, \mathbf{n}\cdot\nabla q = 0 \quad \text{on } \partial\Omega\}.$$

We assume that $\mathbf{f} \in L^2(\Omega)^2$ and define the following least-squares functionals based on (3.5) with $f_3 = f_4 = 0$:

$$G_{-1}(p,\,\mathbf{U};\,\mathbf{f}) = \|\mathbf{f}-\alpha(\nabla\times\mathbf{U})^t - \alpha_2\nabla^\perp(\mathrm{tr}\mathbf{U}) - \nabla^\perp p\|_{-1,0}^2 + \left\|\frac{1}{t}\mathrm{tr}\mathbf{U} - t\Delta p\right\|_{-1}^2 + \|\nabla\cdot\mathbf{U}\|_{-1}^2 \tag{4.1}$$

for $(p,\,\mathbf{U}) \in \mathcal{V}_{-1}$ and

$$G_0(p,\,\mathbf{U};\,\mathbf{f}) = \|\mathbf{f}-\alpha(\nabla\times\mathbf{U})^t - \alpha_2\nabla^\perp(\mathrm{tr}\mathbf{U}) - \nabla^\perp p\|^2 + \left\|\frac{1}{t}\mathrm{tr}\mathbf{U} - t\Delta p\right\|^2 + \|\nabla\cdot\mathbf{U}\|^2 \tag{4.2}$$

for $(p,\,\mathbf{U}) \in \mathcal{V}_0$.

We first establish uniform continuity and ellipticity (i.e., equivalence) of the homogeneous functionals $G_{-1}(p,\,\mathbf{U};\,\mathbf{0})$ and $G_0(p,\,\mathbf{U};\,\mathbf{0})$ in terms of the respective functionals $M_{-1}(p,\,\mathbf{U})$ and $M_0(p,\,\mathbf{U})$ defined on the respective spaces $\mathcal{V}_{-1}$ and $\mathcal{V}_0$ by

$$M_{-1}(p,\mathbf{U}) = \|p\|^2 + \|t\nabla p\|^2 + \left\|\frac{1}{t}\mathrm{tr}\mathbf{U}\right\|_{-1}^2 + \|\mathbf{U}\|^2$$

and

$$M_0(p,\mathbf{U}) = \|p\|_1^2 + \|t\Delta p\|^2 + \left\|\frac{1}{t}\mathrm{tr}\mathbf{U}\right\|^2 + \|\mathbf{U}\|_1^2.$$

THEOREM 4.1. *The functionals* $G_{-1}(p,\mathbf{U};\mathbf{0})$ *and* $M_{-1}(p,\mathbf{U})$ *satisfy the uniform equivalence relation*

$$\frac{1}{C}M_{-1}(p,\mathbf{U}) \le G_{-1}(p,\mathbf{U};\mathbf{0}) \le C\,M_{-1}(p,\mathbf{U}) \tag{4.3}$$

$\forall\ (p,\mathbf{U}) \in \mathcal{V}_{-1}$. *When* (2.3) *holds, the functionals* $G_0(p,\mathbf{U};\mathbf{0})$ *and* $M_0(p,\mathbf{U})$ *satisfy the uniform equivalence relation*

$$\frac{1}{C}M_0(p,\mathbf{U}) \le G_0(p,\mathbf{U};\mathbf{0}) \le C\,M_0(p,\mathbf{U}) \tag{4.4}$$

$\forall\ (p,\mathbf{U}) \in \mathcal{V}_0$.

Proof. The upper bound in (4.3) for G_{-1} follows immediately from the triangle inequality and the easily established bounds

$$\begin{gathered}\|\nabla\times\mathbf{U}\|_{-1,0} \le \|\mathbf{U}\|, \quad \|\nabla^\perp(\mathrm{tr}\mathbf{U})\|_{-1,0} \le \|\mathrm{tr}\mathbf{U}\|, \quad \|\nabla^\perp p\|_{-1,0} \le \|p\|,\\ \|\Delta p\|_{-1} \le \|\nabla p\|, \quad \text{and} \quad \|\nabla\cdot\mathbf{U}\|_{-1} \le \|\mathbf{U}\|.\end{gathered} \tag{4.5}$$

The upper bound in (4.4) for G_0 is a straightforward consequence of the triangle inequality. We first show the validity of the lower bound in (4.4) for the functional G_0. From Theorem 3.1 and the fact that

$$\left\|\frac{1}{t}\mathrm{tr}\mathbf{U}\right\| \le \left\|\frac{1}{t}\mathrm{tr}\mathbf{U} - t\Delta p\right\| + \|t\Delta p\|,$$

it suffices to prove that

$$\|p\|_1^2 + \|t\Delta p\|^2 + \|\nabla\times\mathbf{U}\|^2 \le C\,G_0(p,\mathbf{U};\mathbf{0}) \tag{4.6}$$

$\forall\ (p,\mathbf{U}) \in \mathcal{V}_0$. To this end, assume that the domain is simply connected and that it is a convex polygon or has $C^{1,1}$ boundary. We then have the following Helmholtz decomposition:

$$\mathbf{U} = \nabla\mathbf{u}^t + \nabla^\perp\boldsymbol{\psi}^t, \tag{4.7}$$

where $\mathbf{u} \in (H^1(\Omega)/\mathcal{R})^2$ is the unique solution of

$$(\nabla\mathbf{u},\nabla\mathbf{v}) = (-(\nabla\cdot\mathbf{U})^t,\mathbf{v}) \quad \forall\ \mathbf{v} \in (H^1(\Omega)/\mathcal{R})^2$$

and $\boldsymbol{\psi} \in H_0^1(\Omega)^2$ is the unique solution of

$$(\nabla\boldsymbol{\psi},\nabla\mathbf{v}) = ((\nabla\times\mathbf{U})^t,\mathbf{v}) \quad \forall\ \mathbf{v} \in H_0^1(\Omega)^2.$$

The H^2 regularity of Poisson equations implies that

$$\mathbf{u} \in H^2(\Omega)^2, \quad \boldsymbol{\psi} \in H^2(\Omega)^2, \quad \text{and} \quad \|\mathbf{u}\|_2 \le C\|\Delta\mathbf{u}\|.$$

Now from the H^2 regularity in (2.3) of the perturbed Stokes equations with

$$\mathbf{f} = -\alpha\Delta\boldsymbol{\psi} - \alpha_2\nabla^\perp(\nabla\times\boldsymbol{\psi}) + \nabla^\perp p\ \in L^2(\Omega)^2$$

and

$$f_3 = \frac{1}{t}\nabla\times\boldsymbol{\psi} + t\Delta p - \frac{1}{t}f_4\ \in L^2(\Omega)$$

where $f_4 = \nabla\cdot\mathbf{u} \in H^1(\Omega)$, we then have that

$$\|\boldsymbol{\psi}\|_2 + \|p\|_1 + \|t\Delta p\| \le C\Big(\|\mathbf{f}\| + \|f_3\| + \|\nabla\cdot\mathbf{u}\|_1\Big)$$
$$\le C\Big(\|\mathbf{f} - \alpha_2\nabla^\perp(\nabla\cdot\mathbf{u})\| + \|f_3\| + \|\Delta\mathbf{u}\|\Big). \tag{4.8}$$

Using decomposition (4.7) and $\mathrm{tr}\mathbf{U} = \nabla\cdot\mathbf{u} - \nabla\times\boldsymbol{\psi}$, we note that

$$\|\nabla\times\mathbf{U}\| = \|\Delta\boldsymbol{\psi}\| \le \|\boldsymbol{\psi}\|_2, \quad f_3 = -\frac{1}{t}\mathrm{tr}\mathbf{U} + t\Delta p, \quad \Delta\mathbf{u} = (\nabla\cdot\mathbf{U})^t,$$
$$\text{and}\quad \mathbf{f} - \alpha_2\nabla^\perp(\nabla\cdot\mathbf{u}) = \alpha(\nabla\times\mathbf{U})^t + \alpha_2\nabla^\perp(\mathrm{tr}\mathbf{U}) + \nabla^\perp p,$$

which, together with (4.8), implies (4.6) and, hence, the validity of the lower bound in (4.4) for the functional G_0. The lower bound in (4.3) can be proved in a similar fashion. We have finished the proof of the theorem.

Define the following least-squares functionals based on system (3.3):

$$F_{-1}(\boldsymbol{\phi}, p,\, \mathbf{U};\, \mathbf{f}) = G_{-1}(p,\, \mathbf{U};\, \mathbf{f}) + \|\mathbf{U} - \nabla^\perp\boldsymbol{\phi}^t\|^2$$

for $(\boldsymbol{\phi}, p,\, \mathbf{U}) \in H_0^1(\Omega)^2 \times \mathcal{V}_{-1}$ and

$$F_0(\boldsymbol{\phi}, p,\, \mathbf{U};\, \mathbf{f}) = G_0(p,\, \mathbf{U};\, \mathbf{f}) + \|\mathbf{U} - \nabla^\perp\boldsymbol{\phi}^t\|^2$$

for $(\boldsymbol{\phi}, p,\, \mathbf{U}) \in H_0^1(\Omega)^2 \times \mathcal{V}_0$. Define

$$N_{-1}(\boldsymbol{\phi}, p,\, \mathbf{U}) = M_{-1}(p,\, \mathbf{U}) + \|\boldsymbol{\phi}\|_1^2 \quad\text{and}\quad N_0(\boldsymbol{\phi}, p,\, \mathbf{U}) = M_0(p,\, \mathbf{U}) + \|\boldsymbol{\phi}\|_1^2.$$

COROLLARY 4.1. *The functionals $F_{-1}(\boldsymbol{\phi}, p, \mathbf{U};\, \mathbf{0})$ and $N_{-1}(\boldsymbol{\phi}, p, \mathbf{U})$ satisfy the uniform equivalence relation*

$$\frac{1}{C}N_{-1}(\boldsymbol{\phi}, p, \mathbf{U}) \le F_{-1}(\boldsymbol{\phi}, p, \mathbf{U};\, \mathbf{0}) \le C\, N_{-1}(\boldsymbol{\phi}, p, \mathbf{U}) \tag{4.9}$$

$\forall\ (\boldsymbol{\phi}, p, \mathbf{U}) \in H_0^1(\Omega)^2 \times \mathcal{V}_{-1}$. *When (2.3) holds, the functionals $F_0(\boldsymbol{\phi}, p, \mathbf{U};\, \mathbf{0})$ and $N_0(\boldsymbol{\phi}, p, \mathbf{U})$ satisfy the uniform equivalence relation*

$$\frac{1}{C}N_0(\boldsymbol{\phi}, p, \mathbf{U}) \le F_0(\boldsymbol{\phi}, p, \mathbf{U};\, \mathbf{0}) \le C\, N_0(\boldsymbol{\phi}, p, \mathbf{U}) \tag{4.10}$$

$\forall\ (\boldsymbol{\phi}, p, \mathbf{U}) \in H_0^1(\Omega)^2 \times \mathcal{V}_0$.

Proof. The uniform equivalence relations, (4.9) and (4.10), are immediate consequences of Theorem 4.1 and the triangle inequality.

5. Discrete least-squares functionals. Let $\mathcal{T}_h$ be a partition of the Ω into finite elements; i.e., $\Omega = \cup_{K\in\mathcal{T}_h} K$ with $h = \max\{h_K = \mathrm{diam}(K) : K \in \mathcal{T}_h\}$. Assume that the triangulation $\mathcal{T}_h$ is quasi-uniform; i.e., it is regular and satisfies the inverse assumption (see [17]). Let $\mathcal{P}_1^h$ be a finite-dimensional subspace of $\mathcal{P}_1$ with the following approximation properties:

$$\inf_{q_h\in\mathcal{P}_1^h}\Big(\|q - q_h\| + h\|q - q_h\|_1\Big) \le C\, h^{\gamma_1}\|q\|_{\gamma_1} \quad \forall\ q \in H^{\gamma_1}(\Omega)\cap\mathcal{P}_1, \tag{5.1}$$

$$\inf_{q_h\in\mathcal{P}_1^h}\Big(\sum_{K\in\mathcal{T}_h} h_K\|\Delta(q - q_h)\|_{0,K}\Big) \le C\, h^{\gamma_1-1}\|q\|_{\gamma_1} \quad \forall\ q \in H^{\gamma_1}(\Omega)\cap\mathcal{P}_1, \tag{5.2}$$

where $\gamma_1 \geq 0$ is an integer. It is well known that (5.1) and (5.2) hold for typical finite element spaces consisting of continuous piecewise polynomials with respect to quasi-uniform triangulations (cf. [17]). Let $\mathcal{U}_{-1}^h$ and $\mathcal{U}_0^h$ be the respective finite dimensional subspaces of $H_0(\mathrm{div};\Omega)^2$ and $W \times W$, which will be specified in the subsequent section. Define

$$\mathcal{V}_{-1}^h = \mathcal{P}_1^h \times \mathcal{U}_{-1}^h \quad \text{and} \quad \mathcal{V}_0^h = \mathcal{P}_1^h \times \mathcal{U}_0^h.$$

Note that the functional G_{-1} defined in (4.1) involves the H^{-1} norms, which require solutions of boundary value problems for their evaluations, and the Laplacian operator Δ. Similarly, the functional G_0 defined in (4.2) involves the Laplacian operator. Hence, we need to replace the H^{-1} norms in (4.1) by computationally feasible discrete H^{-1} norms that ensure the equivalence on $\mathcal{V}_{-1}^h$ between the standard norm in $\mathcal{V}_{-1}$ and that induced by the discrete homogeneous functional. (A discrete H^{-1} approach was introduced in [4] for scalar elliptic equations and was extended to the Stokes problem in [12], linear elasticity in [14], and the Reissner–Mindlin plate [15] in the context of least-squares methods and was used for the Stokes problem in [10] in the context of stabilized finite element methods.) Also, we need to replace the Laplacian operator in the second term of the functionals by the corresponding "discrete" operators so that we can use C^0 finite element approximations.

To this end, define the operators A_0: $H_0^{-1}(\Omega) \to H_0^1(\Omega)$ and A: $H^{-1}(\Omega) \to H^1(\Omega)$ as the respective solution operators ($u_0 = A_0 f$ and $u = Af$) for the Poisson problems

$$\text{(5.3)} \qquad \begin{cases} -\Delta u_0 + u_0 &= f \quad \text{in} \quad \Omega, \\ u_0 &= 0 \quad \text{on} \quad \partial\Omega \end{cases}$$

and

$$\text{(5.4)} \qquad \begin{cases} -\Delta u + u &= f \quad \text{in} \quad \Omega, \\ \mathbf{n} \cdot \nabla u &= 0 \quad \text{on} \quad \partial\Omega. \end{cases}$$

It is well known that $(A_0\cdot,\cdot)^{\frac{1}{2}}$ and $(A\cdot,\cdot)^{\frac{1}{2}}$ define norms that are equivalent to the respective H_0^{-1} and H^{-1} norms. Let A_0^h: $H_0^{-1}(\Omega) \to \mathcal{P}_{10}^h$ and A^h: $H^{-1}(\Omega) \to \mathcal{P}_1^h$ be the respective discrete solution operators ($u_0^h = A_0^h f$ and $u^h = A^h f$) for the Poisson problems (5.3) and (5.4) posed on $\mathcal{P}_{10}^h$ and $\mathcal{P}_1^h$:

$$(\nabla u_0^h, \nabla v) + (u_0^h, v) = (f,\, v) \quad \forall\; v \in \mathcal{P}_{10}^h$$

and

$$(\nabla u^h, \nabla v) + (u^h, v) = (f,\, v) \quad \forall\; v \in \mathcal{P}_1^h,$$

where $\mathcal{P}_{10}^h$ is a finite-dimensional subspace of $H_0^1(\Omega)$ satisfying approximation properties (5.1) and (5.2). It is easy to check that $(A_0^h\cdot,\,\cdot)^{\frac{1}{2}}$ and $(A^h\cdot,\,\cdot)^{\frac{1}{2}}$ define the seminorms on the respective $H_0^{-1}(\Omega)$ and $H^{-1}(\Omega)$ which are equivalent to discrete H_0^{-1} and H^{-1} seminorms

$$\|\cdot\|_{-1,0,h} \equiv \sup_{v \in \mathcal{P}_{10}^h} \frac{(\cdot, v)}{\|v\|_1} \quad \text{and} \quad \|\cdot\|_{-1,h} \equiv \sup_{v \in \mathcal{P}_1^h} \frac{(\cdot, v)}{\|v\|_1},$$

respectively. Assume that there are preconditioners B_0^h : $H_0^{-1}(\Omega) \to \mathcal{P}_{10}^h$ and B^h : $H^{-1}(\Omega) \to \mathcal{P}_1^h$ that are symmetric with respect to the $L^2(\Omega)$-inner product and spectrally equivalent to the respective A_0^h and A^h; i.e., there exists a positive constant C, independent of the mesh size h such that

$$\frac{1}{C}(A_0^h v,\, v) \le (B_0^h v,\, v) \le C(A_0^h v,\, v) \quad \forall\, v \in \mathcal{P}_{10}^h \tag{5.5}$$

and that

$$\frac{1}{C}(A^h v,\, v) \le (B^h v,\, v) \le C(A^h v,\, v) \quad \forall\, v \in \mathcal{P}_1^h. \tag{5.6}$$

REMARK 5.1. (1) *By introducing the standard L^2-orthogonal projection operators, it is then easy to check that the spectral equivalences in* (5.5) *and* (5.6) *hold for every* $v \in L^2(\Omega)$*;*

(2) *The spectral equivalences in* (5.5) *and* (5.6) *imply that*

$$|\cdot|_{-1,0,h} \equiv (B_0^h \cdot, \cdot)^{\frac{1}{2}} \quad \textit{and} \quad |\cdot|_{-1,h} \equiv (B^h \cdot, \cdot)^{\frac{1}{2}}$$

define seminorms on $H_0^{-1}(\Omega)$ *and* $H^{-1}(\Omega)$*, which are equivalent to* $\|\cdot\|_{-1,0,h}$ *and* $\|\cdot\|_{-1,h}$*, respectively.*

Finally, we introduce the "discrete" Laplacian operator, Δ_h: $\mathcal{P}_1 \to \mathcal{P}_1^h$, for given $v \in \mathcal{P}_1$ is defined by $u = \Delta_h v \in \mathcal{P}_1^h$ satisfying

$$(u,\, q) = -(\nabla v,\, \nabla q) \quad \forall\, q \in \mathcal{P}_1^h.$$

Now we are ready to define the discrete counterparts of the least-squares functionals G_{-1} and G_0 as follows:

$$\begin{aligned} G_{-1}^h(p, \mathbf{U}; \mathbf{f}) &= |\mathbf{f} - \alpha(\nabla\times\mathbf{U})^t - \alpha_2\nabla^\perp(\mathrm{tr}\mathbf{U}) - \nabla^\perp p|_{-1,0,h}^2 \\ &\quad + \left|\frac{1}{t}\mathrm{tr}\mathbf{U} - t\Delta_h p\right|_{-1,h}^2 + |\nabla\cdot\mathbf{U}|_{-1,h}^2 \\ &\quad + \sum_{K\in\mathcal{T}_h} h_K^2 \|\mathbf{f} - \alpha(\nabla\times\mathbf{U})^t - \alpha_2\nabla^\perp(\mathrm{tr}\mathbf{U}) - \nabla^\perp p\|_{0,K}^2 \\ &\quad + \sum_{K\in\mathcal{T}_h} h_K^2 \left\|\frac{1}{t}\mathrm{tr}\mathbf{U} - t\Delta p\right\|_{0,K}^2 + \sum_{K\in\mathcal{T}_h} h_K^2 \|\nabla\cdot\mathbf{U}\|_{0,K}^2 \end{aligned} \tag{5.7}$$

for any $(p, \mathbf{U}) \in \mathcal{V}_{-1}^h$ and

$$G_0^h(p, \mathbf{U}; \mathbf{f}) = \|\mathbf{f} - \alpha(\nabla\times\mathbf{U})^t - \alpha_2\nabla^\perp(\mathrm{tr}\mathbf{U}) - \nabla^\perp p\|^2 + \left\|\frac{1}{t}\mathrm{tr}\mathbf{U} - t\Delta_h p\right\|^2 + \|\nabla\cdot\mathbf{U}\|^2 \tag{5.8}$$

for $(p,\, \mathbf{U}) \in \mathcal{V}_0^h$, respectively. Let

$$M_0^h(p, \mathbf{U}) = \|p\|_1^2 + \|t\Delta_h p\|^2 + \left\|\frac{1}{t}\mathrm{tr}\mathbf{U}\right\|^2 + \|\mathbf{U}\|_1^2.$$

THEOREM 5.1. *The functionals $G_{-1}^h(p, \mathbf{U};\, \mathbf{0})$ and $M_{-1}(p, \mathbf{U})$ satisfy the uniform equivalence relation*

$$\frac{1}{C} M_{-1}(p, \mathbf{U}) \le G_{-1}^h(p, \mathbf{U};\, \mathbf{0}) \le C\, M_{-1}(p, \mathbf{U}) \tag{5.9}$$

$\forall\ (p,\mathbf{U}) \in \mathcal{V}_0^h$. *When* (2.3) *holds, the functionals* $G_0^h(p,\mathbf{U};\,\mathbf{0})$ *and* $M_0^h(p,\mathbf{U})$ *satisfy the uniform equivalence relation*

$$\frac{1}{C} M_0^h(p,\mathbf{U}) \le G_0^h(p,\mathbf{U};\,\mathbf{0}) \le C\, M_0^h(p,\mathbf{U}) \tag{5.10}$$

$\forall\ (p,\mathbf{U}) \in \mathcal{V}_0^h$.

Proof. The upper bound in (5.9) follows immediately from the triangle and inverse inequalities and the easily established bounds

$$\begin{aligned} &|\nabla\times\mathbf{U}|_{-1,0,h} \le \|\mathbf{U}\|, \quad |\nabla^\perp(\mathrm{tr}\mathbf{U})|_{-1,0,h} \le \|\mathrm{tr}\mathbf{U}\|, \quad |\nabla^\perp p|_{-1,0,h} \le \|p\|, \\ &|\Delta_h p|_{-1,h} \le \|\nabla p\|, \quad \text{and} \quad |\nabla\cdot\mathbf{U}|_{-1,h} \le \|\mathbf{U}\|. \end{aligned} \tag{5.11}$$

To show the validity of the lower bound in (5.9), note first that (see [10] or [15])

$$\|\mathbf{v}\|_{-1,0}^2 \le C\left(\|\mathbf{v}\|_{-1,0,h}^2 + \sum_{K\in\mathcal{T}_h} h_K^2 \|\mathbf{v}\|_{0,K}^2\right) \tag{5.12}$$

and

$$\|\mathbf{v}\|_{-1}^2 \le C\left(\|\mathbf{v}\|_{-1,h}^2 + \sum_{K\in\mathcal{T}_h} h_K^2 \|\mathbf{v}\|_{0,K}^2\right) \tag{5.13}$$

$\forall\ \mathbf{v} \in L^2(\Omega)^2$. It follows from (5.13), the triangle inequality, Remark 5.1, (5.11), and the inverse inequality that

$$\begin{aligned} \left\|\frac{1}{t}\mathrm{tr}\mathbf{U}\right\|_{-1}^2 &\le C\Bigg(\left\|\frac{1}{t}\mathrm{tr}\mathbf{U}\right\|_{-1,h}^2 + \sum_{K\in\mathcal{T}_h} h_K^2 \left\|\frac{1}{t}\mathrm{tr}\mathbf{U}\right\|_{0,K}^2\Bigg) \\ &\le C\Bigg(\left\|\frac{1}{t}\mathrm{tr}\mathbf{U} - t\Delta_h p\right\|_{-1,h}^2 + \|t\Delta_h p\|_{-1,h}^2 + \sum_{K\in\mathcal{T}_h}\left(h_K^2\left\|\frac{1}{t}\mathrm{tr}\mathbf{U} - t\Delta p\right\|_{0,K}^2 + \|t\Delta p\|_{0,K}^2\right)\Bigg) \\ &\le C\Big(G_{-1}^h(p,\mathbf{U};\,\mathbf{0}) + \|t\nabla p\|^2\Big) \end{aligned}$$

and from (2.5), (5.12), the triangle inequality, (5.11), and the inverse inequality that

$$\begin{aligned} \|p\| \le C\|\nabla^\perp p\|_{-1,0}^2 &\le C\left(\|\nabla^\perp p\|_{-1,0,h}^2 + \sum_{K\in\mathcal{T}_h} h_K^2 \|\nabla^\perp p\|_{0,K}^2\right) \\ &\le C\left(G_{-1}^h(p,\mathbf{U};\,\mathbf{0}) + \|\mathbf{U}\|^2\right). \end{aligned}$$

Hence, it suffices to prove that

$$\|t\nabla p\|^2 + \|\mathbf{U}\|^2 \le C\, G_{-1}^h(p,\mathbf{U};\,\mathbf{0}).$$

For any $(p,\mathbf{U}) \in \mathcal{V}_0^h$, since $\mathbf{U} \in H^1(\Omega)^{2^2}$ by Theorem 3.1, it is then sufficient, by (5.12), (5.13), and Remark 5.1, to show that

$$\begin{aligned} \|t\nabla p\|^2 + \|\mathbf{U}\|^2 \le C\Bigg(&\|\alpha(\nabla\times\mathbf{U})^t + \alpha_2\nabla^\perp(\mathrm{tr}\mathbf{U}) + \nabla^\perp p\|_{-1,0}^2 \\ &+ \left\|\frac{1}{t}\mathrm{tr}\mathbf{U} - t\Delta_h p\right\|_{-1,h}^2 + \|\nabla\cdot\mathbf{U}\|_{-1}^2\Bigg). \end{aligned} \tag{5.14}$$

For any $\boldsymbol{\psi} \in H_0^1(\Omega)$ and any $\mathbf{u} \in \mathcal{P}_1 \times \mathcal{P}_1$, using the Poincaré–Friedrichs inequality, integration by parts, identity (3.2), and the definition of the discrete Laplacian operator, we have that

$$\begin{aligned} &C\left(\|\boldsymbol{\psi}\|_1^2 + \|tp\|_1^2\right) \le a(\boldsymbol{\psi},\boldsymbol{\psi}) + \|t\nabla p\|_1^2 \\ &= (-\alpha\Delta\boldsymbol{\psi} - \alpha_2\nabla^\perp(\nabla\times\boldsymbol{\psi}),\boldsymbol{\psi}) - (tp, t\Delta_h p) \\ &= (-\alpha\Delta\boldsymbol{\psi} + \alpha_2\nabla^\perp(\nabla\cdot\mathbf{u} - \nabla\times\boldsymbol{\psi}) + \nabla^\perp p,\boldsymbol{\psi}) + \left(tp, \frac{1}{t}(\nabla\cdot\mathbf{u} - \nabla\times\boldsymbol{\psi}) - t\Delta_h p\right) \\ &\qquad -\alpha_2(\nabla\cdot\mathbf{u},\nabla\times\boldsymbol{\psi}) - (p,\nabla\cdot\mathbf{u}). \end{aligned}$$

Since

$$\|\nabla\cdot\mathbf{u}\| \le \|\mathbf{u}\|_1 \le \|\Delta\mathbf{u}\|_{-1}$$

and

$$\begin{aligned} \|p\| &\le C\|\nabla^\perp p\|_{-1,0} \\ &\le C\left(\|-\alpha\Delta\boldsymbol{\psi} + \alpha_2\nabla^\perp(\nabla\cdot\mathbf{u} - \nabla\times\boldsymbol{\psi}) + \nabla^\perp p\|_{-1,0} + \|\boldsymbol{\psi}\|_1 + \|\Delta\mathbf{u}\|_{-1}\right), \end{aligned}$$

from the definitions of H_0^{-1} norm and H^{-1} seminorm and the Cauchy–Schwarz and ϵ inequalities, we then have that

$$\begin{aligned} \|\boldsymbol{\psi}\|_1^2 + \|tp\|_1^2 \le C\Bigg(&\|-\alpha\Delta\boldsymbol{\psi} + \alpha_2\nabla^\perp(\nabla\cdot\mathbf{u} - \nabla\times\boldsymbol{\psi}) + \nabla^\perp p\|_{-1,0}^2 \\ &+ \left\|\frac{1}{t}(\nabla\cdot\mathbf{u} - \nabla\times\boldsymbol{\psi}) - t\Delta_h p\right\|_{-1,h}^2 + \|\Delta\mathbf{u}\|_{-1}^2\Bigg). \end{aligned}$$

Now (5.14) follows immediately from decomposition (4.7) and the fact that

$$\begin{aligned} &\|t\nabla p\|^2 + \|\mathbf{U}\|^2 = \|t\nabla p\|^2 + \|\nabla\boldsymbol{\psi}\|^2 + \|\nabla\mathbf{u}\|^2 \le \|tp\|_1^2 + \|\boldsymbol{\psi}\|_1^2 + \|\Delta\mathbf{u}\|_{-1}^2 \\ &\le C\Bigg(\|-\alpha\Delta\boldsymbol{\psi} + \alpha_2\nabla^\perp(\nabla\cdot\mathbf{u} - \nabla\times\boldsymbol{\psi}) + \nabla^\perp p\|_{-1,0}^2 \\ &\quad + \left\|\frac{1}{t}(\nabla\cdot\mathbf{u} - \nabla\times\boldsymbol{\psi}) - t\Delta_h p\right\|_{-1,h}^2 + \|\Delta\mathbf{u}\|_{-1}^2\Bigg) \\ &= C\Bigg(\|\alpha(\nabla\times\mathbf{U})^t + \alpha_2\nabla^\perp(\mathrm{tr}\mathbf{U}) + \nabla^\perp p\|_{-1,0}^2 + \left\|\frac{1}{t}\mathrm{tr}\mathbf{U} - t\Delta_h p\right\|_{-1,h}^2 + \|\nabla\cdot\mathbf{U}\|_{-1}^2\Bigg). \end{aligned}$$

This completes the proof of (5.14) and, hence, the lower bound in (5.9).

To show the validity of (5.10), we consider a modification of problem (2.1), i.e., to find $(\boldsymbol{\phi}, p) \in H_0^1(\Omega)^2 \times \mathcal{P}_1^h$ such that

$$\left\{\begin{array}{rcll} a(\boldsymbol{\phi},\boldsymbol{\psi}) + (\nabla^\perp p,\boldsymbol{\psi}) &=& (\mathbf{f},\boldsymbol{\psi}) & \forall\ \boldsymbol{\psi} \in H_0^1(\Omega)^2, \\ (\boldsymbol{\phi},\nabla^\perp q) - t^2(\nabla^\perp p,\nabla^\perp q) &=& (tf_3, q) + (f_4, q) & \forall\ q \in \mathcal{P}_1^h. \end{array}\right.$$

By using the definition of the discrete Laplacian operator, the same proof as that of (2.3) gives the following H^2 regularity estimate:

$$\|\boldsymbol{\phi}\|_2 + \|p\|_1 + \|t\Delta_h p\|^2 \le C\left(\|\mathbf{f}\| + \|f_3\| + \|f_4\|_1\right).$$

Now the proof of (5.10) is similar to that of (4.4). We finish the proof of the theorem.

REMARK 5.2. *Similarly, we can define the discrete counterparts of the functionals* F_k *for* $k = -1, 0$*:*

$$F^h_{-1}(\boldsymbol{\phi}, p, \mathbf{U}) = G^h_{-1}(\boldsymbol{\phi}, p, \mathbf{U}) + \|\mathbf{U} - \nabla^\perp \boldsymbol{\phi}^t\|^2$$

and

$$F^h_0(\boldsymbol{\phi}, p, \mathbf{U}) = G^h_0(\boldsymbol{\phi}, p, \mathbf{U}) + \|\mathbf{U} - \nabla^\perp \boldsymbol{\phi}^t\|^2.$$

It is then immediate consequence of Theorem 5.1 *that*

$$\frac{1}{C} N_{-1}(\boldsymbol{\phi}, p, \mathbf{U}) \le F_{-1}(\boldsymbol{\phi}, p, \mathbf{U}; \mathbf{0}) \le C\, N_{-1}(\boldsymbol{\phi}, p, \mathbf{U})$$

and that

$$\frac{1}{C} N^h_0(\boldsymbol{\phi}, p, \mathbf{U}) \le F_0(\boldsymbol{\phi}, p, \mathbf{U}; \mathbf{0}) \le C\, N^h_0(\boldsymbol{\phi}, p, \mathbf{U})$$

$\forall\ (\boldsymbol{\phi}, p, \mathbf{U}) \in H^1_0(\Omega)^2 \times \mathcal{V}^h_0$, *where* $N^h_0(\boldsymbol{\phi}, p, \mathbf{U}) = M^h_0(\boldsymbol{\phi}, p, \mathbf{U}) + \|\boldsymbol{\phi}\|^2_1$. *Since the functionals* F^h_k *involve more independent variables than those of the functionals* G^h_k*, we will consider two-stage algorithms based on the functionals* G^h_k *in the subsequent section.*

6. Finite element approximations.

Similar to discussions for linear elasticity in [14], the expression $tr\mathbf{U} = U_{11} + U_{22}$ in the functionals G^h_k $(k = -1, 0)$ represents an intimate coupling between U_{11} and U_{22}. When t is small, this coupling must tend to dominate. This causes degrading approximation properties of the discretization and convergence properties of the solution process, but it is eliminated by a simple rotation applied to $\mathbf{U}$. This section first describes finite element approximations based on the respective functionals G^h_k for $k = -1, 0$ and then establishes optimal order error estimates.

To this end, we first rearrange the matrix $\mathbf{U}$ as a 4×1 column vector

$$\mathbf{U} = (U_{ij})_{2\times 2} = (U_{11}, U_{21}, U_{12}, U_{22})^t = (U_1, U_2, U_3, U_4)^t.$$

We apply a rotation matrix R to the column vector $\mathbf{U}$ designed in part so that the first component of $R\mathbf{U}$ is proportional to $\mathrm{tr}\mathbf{U} = U_{11} + U_{22}$. The rotation matrix we use here is defined by

$$R = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 & -\frac{1}{\sqrt{2}} \end{pmatrix}, \tag{6.1}$$

which is a symmetric and orthogonal matrix. It is easy to see that

$$\mathbf{V} = R\mathbf{U} = \left(\frac{1}{\sqrt{2}} \mathrm{tr}\mathbf{U}, U_{21}, U_{12}, \frac{1}{\sqrt{2}}(U_{11} - U_{22}) \right)^t.$$

Define the space $\tilde{\mathcal{U}} \equiv R(W \times W) = \{\mathbf{V} = R\mathbf{U} : \mathbf{U} \in W \times W\}$. Note that $W \times W = R\tilde{\mathcal{U}}$ and that each vector $\mathbf{U} \in W \times W$ is of the form

$$\mathbf{U} = R\mathbf{V}, \quad \mathbf{V} \in \tilde{\mathcal{U}}.$$

Note also that spaces $W \times W$ and $\tilde{\mathcal{U}}$ are the same up to boundary conditions. Let $\tilde{\mathcal{U}}^h$ be a finite-dimensional subspace of $\tilde{\mathcal{U}}$. Assume that it satisfies the following approximation property: there exist a constant C and an integer $\gamma_2 \geq 0$ such that for all $\mathbf{V} \in H^{\gamma_2}(\Omega)^4 \cap \tilde{\mathcal{U}}$, there exists $\tilde{\mathbf{V}}^h \in \tilde{\mathcal{U}}^h$ such that

$$h^{-1}\|V_j - \tilde{V}_j^h\|_{-1} + \|V_j - \tilde{V}_j^h\| + h\|V_j - \tilde{V}_j^h\|_1 \leq C\, h^{\gamma_2}\|V_j\|_{\gamma_2}, \quad j = 1, 2, 3, 4. \tag{6.2}$$

It is easy to see that (6.2) holds for Lagrange finite element spaces consisting of continuous piecewise polynomials. Note that the boundary conditions on $\tilde{\mathcal{U}}^h$ can be implemented for polygonal domains by imposing simple algebraic relations on the boundary nodes.

Least-squares finite element approximations to the solution $(p, \mathbf{U})$ of system (3.5–3.6) are defined by either of the following two algorithms:

- *the discrete H^{-1} approach*: Let $(p^h, \mathbf{V}^h) \in \mathcal{P}_1^h \times \tilde{\mathcal{U}}^h$ be the unique solution of

$$G_{-1}^h(p^h, R\mathbf{V}^h; \mathbf{f}) = \min\{G_{-1}^h(q, R\mathbf{W}; \mathbf{f}) : \ (q, \mathbf{W}) \in \mathcal{P}_1^h \times \tilde{\mathcal{U}}^h\} \tag{6.3}$$

  and set $(p^h, \mathbf{U}^h) = (p^h, R\mathbf{V}^h)$.
- *the L^2 approach*: Let $(p^h, \mathbf{V}^h) \in \mathcal{P}_1^h \times \tilde{\mathcal{U}}^h$ be the unique solution of

$$G_0^h(p^h, R\mathbf{V}^h; \mathbf{f}) = \min\{G_0^h(q, R\mathbf{W}; \mathbf{f}) : \ (q, \mathbf{W}) \in \mathcal{P}_1^h \times \tilde{\mathcal{U}}^h\} \tag{6.4}$$

  and set $(p^h, \mathbf{U}^h) = (p^h, R\mathbf{V}^h)$.

If the rotation, $\boldsymbol{\phi}$, is desired, its approximation can be computed by solving two discrete Poisson equations, i.e., find $\boldsymbol{\phi}^h \in \mathcal{P}_{10}^h \times \mathcal{P}_{10}^h$ that satisfies

$$\|\nabla\boldsymbol{\phi}^h - \mathbf{U}^h\| = \min\{\|\nabla\boldsymbol{\psi} - \mathbf{U}^h\| \,:\, \boldsymbol{\psi} \in \mathcal{P}_{10}^h \times \mathcal{P}_{10}^h\}. \tag{6.5}$$

Let

$$\tilde{M}_{-1}^h(p, \mathbf{V}) = \|p\|^2 + \|t\nabla p\|^2 + \left\|\frac{1}{t}V_1\right\|_{-1}^2 + \|\mathbf{V}\|^2$$

and

$$\tilde{M}_0^h(p, \mathbf{V}) = \|p\|_1^2 + \|t\Delta_h p\|^2 + \left\|\frac{1}{t}V_1\right\|^2 + \|\mathbf{V}\|_1^2,$$

where V_1 is the first component of the 4×1 column vector $\mathbf{V}$.

COROLLARY 6.1. *For all $(p, \mathbf{V}) \in \mathcal{P}_1^h \times \tilde{\mathcal{U}}^h$, the functionals $G_{-1}^h(p, R\mathbf{V}; \mathbf{0})$ and $\tilde{M}_{-1}^h(p, \mathbf{V})$ satisfy the uniform equivalence relation*

$$\frac{1}{C}\tilde{M}_{-1}^h(p, \mathbf{V}) \leq G_{-1}^h(p, R\mathbf{V}; \mathbf{0}) \leq C\tilde{M}_{-1}^h(p, \mathbf{V}); \tag{6.6}$$

if (2.3) holds, the functionals $G_0^h(p, R\mathbf{V}; \mathbf{0})$ and $\tilde{M}_0^h(p, \mathbf{V})$ satisfy the uniform equivalence relation

$$\frac{1}{C}\tilde{M}_0^h(p, \mathbf{V}) \leq G_0^h(p, R\mathbf{V}; \mathbf{0}) \leq C\tilde{M}_0^h(p, \mathbf{V}). \tag{6.7}$$

Let $(p, \mathbf{U}) \in \mathcal{V}_{-1}$ be the solution of (3.5)–(3.6) with $f_3 = f_4 = 0$ and $(p^h, \mathbf{V}^h) \in \mathcal{P}_1^h \times \tilde{\mathcal{U}}^h$ the solution of (6.3), and let $\mathbf{U}^h = R\mathbf{V}^h$. Let $b_{-1}^h(\cdot\,;\,\cdot)$ denote the bilinear form induced by the quadratic form $G_{-1}^h(\cdot\,;\mathbf{0})$, it is then easy to check the following orthogonality property:

$$b_{-1}^h(p - p^h, R\mathbf{V} - R\mathbf{V}^h;\, q, R\mathbf{W}) = 0 \quad \forall\ (q, \mathbf{W}) \in \mathcal{P}_1^h \times \tilde{\mathcal{U}}^h, \tag{6.8}$$

where $\mathbf{V} = R\mathbf{U}$.

THEOREM 6.1. *Assume that* $(p, \mathbf{U})$ *is in* $H^{\gamma+1}(\Omega) \times H^{\gamma}(\Omega)^{2^2}$ *with* $\gamma \geq 1$. *Then there exists a positive constant* C *independent of the thickness* t *and the mesh size* h *such that*

$$\begin{aligned} &\|p - p^h\| + \|t(p - p^h)\|_1 + \left\|\frac{1}{t}\mathrm{tr}(\mathbf{U} - \mathbf{U}^h)\right\|_{-1} + \|\mathbf{U} - \mathbf{U}^h\| \\ &\leq Ch^{\gamma}\left(\|p\|_{\gamma} + \|tp\|_{\gamma+1} + \left\|\frac{1}{t}\mathrm{tr}\mathbf{U}\right\|_{\gamma-1} + \|\mathbf{U}\|_{\gamma}\right). \end{aligned} \tag{6.9}$$

Moreover, if $\gamma = 1$, *we then have that*

$$\|p - p^h\| + \|t(p - p^h)\|_1 + \left\|\frac{1}{t}\mathrm{tr}(\mathbf{U} - \mathbf{U}^h)\right\|_{-1} + \|\mathbf{U} - \mathbf{U}^h\| \leq Ch\|\mathbf{f}\|. \tag{6.10}$$

Proof. Let $\mathbf{V} = R\mathbf{U}$, by the relations that $\mathbf{U}^h = R\mathbf{V}^h$, $\mathrm{tr}\mathbf{U} = \frac{1}{\sqrt{2}}V_1$, and $\mathrm{tr}\mathbf{U}^h = \frac{1}{\sqrt{2}}V_1^h$; to show the validity of the error estimate in (6.9), it suffices to prove that

$$\tilde{M}_{-1}^h(p - p^h, \mathbf{V} - \mathbf{V}^h) \leq Ch^{2\gamma}\left(\|p\|_{\gamma}^2 + \|tp\|_{\gamma+1}^2 + \left\|\frac{1}{t}V_1\right\|_{\gamma-1}^2 + \|\mathbf{V}\|_{\gamma}^2\right). \tag{6.11}$$

Let $p^I \in \mathcal{P}_1^h$ be an interpolant of p; it then follows from (5.1) with $\gamma_1 = \gamma$ and $\gamma + 1$ and (6.2) with $\gamma_2 = \gamma - 1$ that

$$\tilde{M}_{-1}^h(p - p^I, \mathbf{V} - \tilde{\mathbf{V}}^h) \leq Ch^{2\gamma}\left(\|p\|_{\gamma}^2 + \|tp\|_{\gamma+1}^2 + \left\|\frac{1}{t}V_1\right\|_{\gamma-1}^2 + \|\mathbf{V}\|_{\gamma}^2\right). \tag{6.12}$$

Uniform equivalence relation (6.6), orthogonality (6.8), and the Cauchy–Schwarz inequality imply

$$\begin{aligned} &\frac{1}{C}\tilde{M}_{-1}^h(p^I - p^h, \tilde{\mathbf{V}}^h - \mathbf{V}^h) \\ &\leq b_{-1}^h(p^I - p^h, R(\tilde{\mathbf{V}}^h - \mathbf{V}^h); p^I - p^h, R(\tilde{\mathbf{V}}^h - \mathbf{V}^h)) \\ &= b_{-1}^h(p^I - p, R(\tilde{\mathbf{V}}^h - \mathbf{V}); p^I - p^h, R(\tilde{\mathbf{V}}^h - \mathbf{V}^h)) \\ &\leq C\left(G_{-1}^h(p^I - p, R(\tilde{\mathbf{V}}^h - \mathbf{V}); \mathbf{0})\right)^{\frac{1}{2}}\left(\tilde{M}_{-1}^h(p^I - p^h, \tilde{\mathbf{V}}^h - \mathbf{V}^h)\right)^{\frac{1}{2}}. \end{aligned}$$

Hence,

$$\tilde{M}_{-1}^h(p^I - p^h, \tilde{\mathbf{V}}^h - \mathbf{V}^h) \leq C\, G_{-1}^h\left(p^I - p, R(\tilde{\mathbf{V}}^h - \mathbf{V}); \mathbf{0}\right). \tag{6.13}$$

From the triangle inequality, (5.11), approximation properties (5.1) with $\gamma_1 = \gamma$ and $\gamma + 1$, (6.2) with $\gamma_2 = \gamma$ and $\gamma - 1$, and (5.2) with $\gamma_1 = \gamma$, we have that

$$
\begin{aligned}
&G_{-1}^h\left(p - p^I, R(\mathbf{V} - \tilde{\mathbf{V}}^h); \mathbf{0}\right) \\
&\le C\Bigg(\|\mathbf{V} - \tilde{\mathbf{V}}^h\|^2 + \|p - p^I\|^2 + \left\|\frac{1}{t}(V_1 - \tilde{V}_1^h)\right\|_{-1}^2 + \|t(p - p^I)\|_1^2 \\
&+ \sum_{K\in\mathcal{T}_h} h_K^2 \left(\|\mathbf{V} - \tilde{\mathbf{V}}^h\|_{1,K}^2 + \|p - p^I\|_{1,K}^2 + \left\|\frac{1}{t}(V_1 - \tilde{V}_1^h)\right\|_{0,K}^2 + \|t\Delta(p - p^I)\|_{0,K}^2\right)\Bigg) \\
&\le Ch^{2\gamma}\left(\|p\|_\gamma^2 + \|tp\|_{\gamma+1}^2 + \left\|\frac{1}{t}V_1\right\|_{\gamma-1}^2 + \|\mathbf{V}\|_\gamma^2\right).
\end{aligned}
$$

Combining with (6.13), (6.12), and the triangle inequality, we finish the proof of (6.11) and, hence, the error bound in (6.9).

If $\gamma = 1$, then the second equation in (3.5) and (2.3) in Theorem 2.1 with $f_3 = f_4 = 0$ imply that

$$\left\|\frac{1}{t}\mathrm{tr}\mathbf{U}\right\| = \|t\Delta p\| \le C\|\mathbf{f}\|.$$

Now (6.10) is then a direct consequence of (6.9), Theorem 2.1 with $f_3 = f_4 = 0$, and the relation $\mathbf{U} = \nabla^\perp \boldsymbol{\phi}$. This completes the proof of the theorem.

Let $(p, \mathbf{U}) \in \mathcal{V}_0$ be the solution of (3.5)–(3.6) with $f_3 = f_4 = 0$ and $(p^h, \mathbf{V}^h) \in \mathcal{P}_1^h \times \tilde{\mathcal{U}}^h$ the solution of (6.4), and let $\mathbf{U}^h = R\mathbf{V}^h$. Let $b_0^h(\cdot\,;\,\cdot)$ denote the bilinear form induced by the quadratic form $G_0^h(\cdot\,;\mathbf{0})$; it is then easy to check the following orthogonality property:

$$b_0^h(p - p^h, R\mathbf{V} - R\mathbf{V}^h;\, q, R\mathbf{W}) = 0 \quad \forall\ (q, \mathbf{W}) \in \mathcal{P}_1^h \times \tilde{\mathcal{U}}^h, \tag{6.14}$$

where $\mathbf{V} = R\mathbf{U}$.

THEOREM 6.2. *Assume that $(p, \mathbf{U})$ is in $H^{\gamma+1}(\Omega) \times H^{\gamma+1}(\Omega)^{2^2}$ with $\gamma \ge 1$. Then there exists a positive constant C independent of the thickness t and the mesh size h such that*

$$\|p - p^h\|_1 + \left\|\frac{1}{t}\mathrm{tr}(\mathbf{U} - \mathbf{U}^h)\right\| + \|\mathbf{U} - \mathbf{U}^h\|_1 \le Ch^\gamma\left(\|p\|_{\gamma+1} + \left\|\frac{1}{t}\mathrm{tr}\mathbf{U}\right\|_\gamma + \|\mathbf{U}\|_{\gamma+1}\right). \tag{6.15}$$

Proof. Let $\mathbf{V} = R\mathbf{U}$ and $p^I \in \mathcal{P}_1^h$ be an interpolant of p, it then follows from (5.1) and (6.2) that

$$\|p - p^I\|_1 + \left\|\frac{1}{t}(V_1 - \tilde{V}_1^h)\right\| + \|\mathbf{V} - \tilde{\mathbf{V}}^h\|_1 \le Ch^\gamma\left(\|p\|_{\gamma+1} + \left\|\frac{1}{t}V_1\right\|_\gamma + \|\mathbf{V}\|_{\gamma+1}\right).$$

The rest of the proof is the same as that in the proof of Theorem 6.1.

Equivalence (6.6) in Corollary 6.1 indicates that the discrete functional G_{-1}^h $(p, R\mathbf{V}; \mathbf{0})$ is uniformly equivalent to the simple functional $\tilde{M}_{-1}^h(p, R\mathbf{V})$ on $\mathcal{P}_1^h \times \tilde{\mathcal{U}}^h$.

Thus, rescaling the first component of vector $\mathbf{V}$ by $\frac{1}{t}$, i.e., $\bar{V}_1 = \frac{1}{t}V_1$, makes G^h_{-1} uniformly equivalent to $\|p\|^2 + \|t\nabla p\|^2 + \|\bar{V}_1\|^2_{-1} + \|t\bar{V}_1\|^2 + \sum_{i=2}^4 \|V_i\|^2$. We can then use any effective elliptic preconditioners associated with p, including those of multigrid type, and simple preconditioners associated with $\bar{V}_1$ and V_i for $i = 2, 3, 4$, including those of diagonal matrix type. When t is relatively small compared to the mesh size h, equivalence (6.7) implies that standard multigrid solution methods for minimizing $G^h_0(p^h, R\mathbf{V}^h; \mathbf{f})$ converge uniformly in t and h.

7. The Reissner–Mindlin plate. In previous sections, we developed least-squares approaches for the perturbed Stokes equations and, hence, for the Reissner–Mindlin plate through the three-stage algorithm by Brezzi and Fortin in [6]. In this section, we develop a direct least-squares approach for the Reissner–Mindlin plate by computing the *transverse shear strain* and the rotation flux. If the rotation $\boldsymbol{\phi}$ and the transverse displacement ω are desired, they may be recovered by solving first two Poisson equations for $\boldsymbol{\phi}$ and then one Poisson equation for ω. Because the development of computable finite element approximations and the corresponding iterative solvers or preconditioners, based on the least-squares functional involving H^{-1} norms, becomes standard (see, for example, sections 5 and 6), we are, therefore, focusing on establishing ellipticity and continuity here.

We introduce the rotation flux $\mathbf{U} = \nabla^\perp \boldsymbol{\phi}^t$ as in section 3 and the transverse shear strain

$$\boldsymbol{\eta} = \frac{1}{t^2}\left(\boldsymbol{\phi} - \nabla\omega\right);$$

the definition of $\boldsymbol{\eta}$ and the homogeneous Dirichlet boundary conditions of $\boldsymbol{\phi}$ and ω imply that

$$\nabla\times\boldsymbol{\eta} = \frac{1}{t^2}\nabla\times\boldsymbol{\phi} = -\frac{1}{t^2}\mathrm{tr}\mathbf{U} \quad \text{in } \Omega \quad \text{and} \quad \boldsymbol{\tau}\cdot\boldsymbol{\eta} = 0 \quad \text{on } \partial\Omega. \tag{7.1}$$

Hence, a reduced system to determine $(\mathbf{U}, \boldsymbol{\eta})$ for the Reissner–Mindlin plate in (1.1–1.2) has of the form

$$\left\{ \begin{array}{rcll} \alpha(\nabla\times\mathbf{U})^t + \alpha_2\nabla^\perp(\mathrm{tr}\mathbf{U}) + \boldsymbol{\eta} & = & \mathbf{0} & \text{in} \quad \Omega, \\ \nabla\cdot\boldsymbol{\eta} & = & g & \text{in} \quad \Omega, \\ \mathrm{tr}\mathbf{U} + t^2\nabla\times\boldsymbol{\eta} & = & 0 & \text{in} \quad \Omega, \\ \nabla\cdot\mathbf{U} & = & \mathbf{0}^t & \text{in} \quad \Omega \end{array} \right. \tag{7.2}$$

with boundary conditions

$$\mathbf{n}\cdot\mathbf{U} = \mathbf{0}^t \quad \text{and} \quad \boldsymbol{\tau}\cdot\boldsymbol{\eta} = 0 \quad \text{on } \partial\Omega. \tag{7.3}$$

Define the least-squares functional as follows:

$$G(\mathbf{U}, \boldsymbol{\eta};\, g) = \|\alpha(\nabla\times\mathbf{U})^t + \alpha_2\nabla^\perp(\mathrm{tr}\mathbf{U}) + \boldsymbol{\eta}\|^2 + \|g - \nabla\cdot\boldsymbol{\eta}\|^2_{-1,0} + \left\|\frac{1}{t}\mathrm{tr}\mathbf{U} + t\nabla\times\boldsymbol{\eta}\right\|^2 + \|\nabla\cdot\mathbf{U}\|^2. \tag{7.4}$$

Let

$$M(\mathbf{U}, \boldsymbol{\eta}) = \left\|\frac{1}{t}\mathrm{tr}\mathbf{U}\right\|^2 + \|\mathbf{U}\|_1^2 + \|\boldsymbol{\eta}\|^2 + \|t\nabla\times\boldsymbol{\eta}\|^2.$$

THEOREM 7.1. *The functionals $G(\mathbf{U},\boldsymbol{\eta};0)$ and $M(\mathbf{U},\boldsymbol{\eta})$ satisfy the uniform equivalence relation*

$$\frac{1}{C}M(\mathbf{U},\boldsymbol{\eta}) \le G(\mathbf{U},\boldsymbol{\eta};\,0) \le C\,M(\mathbf{U},\boldsymbol{\eta}) \tag{7.5}$$

$\forall\ (\mathbf{U},\boldsymbol{\eta}) \in (W\times W)\times H_0(\mathrm{curl}\,;\Omega)$.

Proof. The upper bound in (7.5) for G is an immediate consequence of the triangle inequality and the easily established bound

$$\|\nabla\cdot\boldsymbol{\eta}\|_{-1,0} \le \|\boldsymbol{\eta}\|.$$

To show the validity of the lower bound in (7.5), we use the following Helmholtz decomposition:

$$\boldsymbol{\eta} = \nabla q + \nabla^{\perp} p, \tag{7.6}$$

where $q \in H_0^1(\Omega)$ is the unique solution of

$$(\nabla q, \nabla s) = (-\nabla\cdot\boldsymbol{\eta}, s) \quad \forall\ s \in H_0^1(\Omega)$$

and $p \in H^1(\Omega)/\mathcal{R}$ is the unique solution of

$$(\nabla p, \nabla s) = (\nabla\times\boldsymbol{\eta}, s) \quad \forall\ s \in H^1(\Omega)/\mathcal{R}.$$

It follows from decomposition (7.6), the lower bound in (4.4), and the triangle inequality that

$$\begin{aligned}
&\|t\nabla\times\boldsymbol{\eta}\|^2 + \left\|\frac{1}{t}\mathrm{tr}\mathbf{U}\right\|^2 + \|\mathbf{U}\|_1^2 \\
&= \|t\Delta p\|^2 + \left\|\frac{1}{t}\mathrm{tr}\mathbf{U}\right\|^2 + \|\mathbf{U}\|_1^2 \le C\,G_0(p,\mathbf{U};\mathbf{0}) \\
&\le C\Bigg(\|\alpha(\nabla\times\mathbf{U})^t + \alpha_2\nabla^{\perp}(\mathrm{tr}\mathbf{U}) + \nabla^{\perp}p + \nabla q\|^2 + \|\Delta q\|_{-1,0}^2 \\
&\qquad + \left\|\frac{1}{t}\mathrm{tr}\mathbf{U} - t\Delta p\right\|^2 + \|\nabla\cdot\mathbf{U}\|^2\Bigg) \\
&= C\,G(\mathbf{U},\boldsymbol{\eta};\,0).
\end{aligned}$$

Now the lower bound in (7.5) for the term $\|\boldsymbol{\eta}\|^2$ is a direct consequence of the triangle inequality. This completes the proof of lower bound (7.5) and, hence, the theorem.

The solution $(\boldsymbol{\phi},\omega)$ of the Reissner–Mindlin plate (1.1)–(1.2) can be obtained by the following three-stage algorithm.

Stage 1: Let $(\mathbf{V},\boldsymbol{\eta}) \in \tilde{\mathcal{U}} \times H_0(\mathrm{curl}\,;\Omega)$ be the unique solution of

$$G(R\mathbf{V},\boldsymbol{\eta};\,g) = \min\{G(R\mathbf{W},\boldsymbol{\xi};\,g) :\ (\mathbf{W},\boldsymbol{\xi}) \in \tilde{\mathcal{U}} \times H_0(\mathrm{curl}\,;\Omega)\} \tag{7.7}$$

and set $(\mathbf{U},\boldsymbol{\eta}) = (R\mathbf{V},\boldsymbol{\eta})$.

Stage 2: Let $\boldsymbol{\phi} \in H_0^1(\Omega)^2$ be the unique solution of

$$\|\nabla\boldsymbol{\phi} - \mathbf{U}\| = \min\{\|\nabla\boldsymbol{\psi} - \mathbf{U}\|\ :\ \boldsymbol{\psi} \in H_0^1(\Omega)^2\}. \tag{7.8}$$

Stage 3: Let $\omega \in H_0^1(\Omega)$ be the unique solution of

$$\|\nabla\omega - (\boldsymbol{\phi} - t^2\boldsymbol{\eta})\| = \min\{\|\nabla\sigma - (\boldsymbol{\phi} - t^2\boldsymbol{\eta})\|\ :\ \sigma \in H_0^1(\Omega)\}. \tag{7.9}$$

REMARK 7.1. *Note that the homogeneous functional* $G(\mathbf{U}, \boldsymbol{\eta}; 0)$ *is uniformly equivalent to the functional* $\|\frac{1}{t}\mathrm{tr}\mathbf{U}\|^2 + \|\mathbf{U}\|_1^2 + \|\boldsymbol{\eta}\|^2 + \|t\nabla\times\boldsymbol{\eta}\|^2$. *Hence, standard finite element approximations give optimal order error estimates uniformly in* t. *But standard multigrid methods do not converge uniformly in* t *and* h *since the operator* $I - t^2\nabla^\perp\nabla\times$ *has high frequency eigenfunctions corresponding to the eigenvalue* 1. *If we use finite element subspaces in* $H_0(\mathrm{curl}\,;\Omega)$ *for* $\boldsymbol{\eta}$, *then the multigrid method with the Schwarz alternating procedure as the smoother converges uniformly in* t *and* h (*see* [3]). *If* t *is relative small compared to the mesh size* h, *then a simple iteration method like Jacobi for* $\boldsymbol{\eta}$ *also converges uniformly in* t *and* h.

REFERENCES

[1] D. N. Arnold and R. S. Falk , *A uniformly accurate finite element method for the Reissner-Mindlin plate*, SIAM J. Numer. Anal., 26 (1989), pp. 1276–1290.

[2] D. N. Arnold, R. S. Falk, and R. Winther, *Preconditioning discrete approximations of the Reissner-Mindlin plate model*, RAIRO Modél. Math. Anal. Numér., 31 (1997), pp. 517–557.

[3] D. N. Arnold, R. S. Falk, and R. Winther, *Multigrid in H(div) and H(curl)*, Numer. Math., 85 (2000), pp. 197–218.

[4] J. H. Bramble, R. D. Lazarov, and J. E. Pasciak, *A least-squares approach based on a discrete minus one inner product for first order system*, Math. Comp., 66 (1997), pp. 935–955.

[5] J. H. Bramble and T. Sun, *A negative-norm least squares method for Mindlin-Reissner plates*, Math. Comp., 67 (1998), pp. 901–916.

[6] F. Brezzi and M. Fortin, *Numerical approximation of Mindlin-Reissner plates*, Math. Comp., 47 (1986), pp. 151–158.

[7] F. Brezzi, K. Bathe, and M. Fortin, *Mixed interpolated elements for Reissner-Mindlin plates*, Internat. J. Numer. Methods Engrg., 28 (1989), pp. 1787–1801.

[8] F. Brezzi, M. Fortin, and R. Stenberg, *Error analysis of mixed-interpolated elements for Mindlin-Reissner plates*, Math. Models Methods Appl. Sci., 1 (1991), pp. 125–151.

[9] F. Brezzi, L. Franca, T. Hughes, and A. Russo, *Stabilization Techniques and Subgrid Scales Capturing*, UCD/CCM Report 83, University of Colorado at Denver, 1996.

[10] Z. Cai and J. Douglas, Jr., *Stabilized finite element methods with fast iterative solution algorithms for the Stokes problem*, Comput. Methods Appl. Mech. Engrg., 166 (1998), pp. 115-129.

[11] Z. Cai, C.-O. Lee, T. A. Manteuffel, and S. F. McCormick, *First-order system least squares for the Stokes and linear elasticity equations: Further results*, SIAM J. Sci. Comput., 21 (2000), pp. 1728–1739.

[12] Z. Cai, T. Manteuffel, and S. McCormick, *First-order system least squares for velocity-vorticity-pressure form of the Stokes equations, with application to linear elasticity*, Electron. Trans. Numer. Anal., 3 (1995), pp. 150–159.

[13] Z. Cai, T. A. Manteuffel, and S. F. McCormick, *First-order system least squares for the Stokes equations, with application to linear elasticity*, SIAM J. Numer. Anal., 34 (1997), pp. 1727–1741.

[14] Z. Cai, T. A. Manteuffel, S. F. McCormick, and S. V. Parter, *First-order system least squares (FOSLS) for planar linear elasticity: Pure traction problem*, SIAM J. Numer. Anal., 35 (1998), pp. 320–335.

[15] Z. Cai, X. Ye, and H. Zhang, *Least-squares finite element approximations for the Reissner-Mindlin plate*, Numer. Linear Algebra Appl., 6 (1999), pp. 479–496.

[16] D. Chapelle and R. Stenberg, *An optimal low-order locking free finite element method for Reissner-Mindlin plates*, Math. Model. Methods Appl. Sci., 8 (1998), pp. 407–430.

[17] P. G. Ciarlet, *The Finite Element Method for Elliptic Problems*, North-Holland, New York, 1978.

[18] V. Girault and P. A. Raviart, *Finite Element Methods for Navier-Stokes Equations: Theory and Algorithms*, Springer-Verlag, New York, 1986.

[19] R. B. Kellogg and J. E. Osborn, *A regularity result for the Stokes problem in a convex polygon*, J. Funct. Anal., 21 (1976), pp. 397–431.

[20] O. A. Ladyzhenskaya, *The Mathematical Theory of Viscous Incompressible Flows*, Gordon and Breach, New York, 1963.