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SOLUTION METHODS FOR THE POISSON EQUATION WITH CORNER SINGULARITIES: NUMERICAL RESULTS*

ZHIQIANG CAI†, SEOKCHAN KIM‡, AND BYEONG-CHUN SHIN§

Abstract. In [Z. Cai and S. Kim, *SIAM J. Numer. Anal.,* 39 (2001), pp. 286–299], we developed a new finite element method using singular functions for the Poisson equation on a polygonal domain with re-entrant angles. Such a method first computes the regular part of the solution, then the stress intensity factor, and finally the solution itself. This paper studies solution methods for solving the system of linear equations arising from the discretization and focuses on numerical results including the finite element accuracy and the multigrid performance.

Key words. corner singularity, finite element, multigrid

AMS subject classifications. 65F10, 65F30, 65N55

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1. Introduction. Solutions of elliptic boundary value problems on a domain with corners have singular behavior near the corners. This occurs even when data of the underlying problem are very smooth. Such singular behavior affects the accuracy of the finite element method throughout the whole domain. In [5], we developed a new finite element method using singular and dual singular functions for the Poisson equation with homogeneous Dirichlet boundary conditions on a polygonal domain with re-entrant angles. By using the dual singular function and a cut-off function with a bigger support, we derived a new extraction formula for the stress intensity factor λ in terms of the regular part w of the solution u. This enables us to deduce a well-posed problem for w, which is then approximated by the continuous piecewise linear finite element method. Approximations to the intensity factor λ and the solution u are straightforward, so we concentrate in this paper on the computation of w. It was shown in [5] that we achieve $O(h)$ optimal accuracy for w and u in the H^1 norm. We established the $O(h^{1+\frac{\pi}{\omega}})$ error bound for w in L^2, which, in turn, implies the same error bounds for u in L^2 and for λ in the absolute value, where ω is the internal angle. This error bound for w in L^2 is not optimal. The reason is the simplified adjoint problem used in the duality argument, which does not have full regularity. But w is H^2 regular; it is then interesting to see if such error bound is sharp numerically.

The problem determining w is no longer the nice Poisson equation but is perturbed by integral terms which are only nonzero on a strip away from the corner. Because of such a perturbation, the problem is nonsymmetric and probably indefinite. To solve the nonsymmetric algebraic equations arising from the discretization, we observe that the perturbation is rank one and that its pseudodifferential order is -1. The

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†Department of Mathematics, Purdue University, 1395 Mathematical Science Building, West Lafayette, IN 47907-1395 (zcai@math.purdue.edu). This author was sponsored in part by the National Science Foundation under grants DMS-9619792 and INT-9910010.

‡Department of Applied Mathematics, Changwon National University, Changwon, 641-773, Republic of Korea (sckim@sarim.changwon.ac.kr). This author was supported by the basic research program of the Korea Science and Engineering Foundation under grant KOSEF 1999-2-103-0002-3.

§Department of Mathematics, Ajou University, Suwon 442-749, Korea (bcshin@math.purdue.edu). This auther was supported by the postdoctoral fellowships program of the Korea Science and Engineering Foundation (KOSEF).

former leads to the first approach which uses the Sherman–Morrison (SM) formula (see section 4.1). This approach requires *two* (approximate) inversions of the discrete Laplace operator, which will be done by multigrid (MG) methods. It is well known (see, e.g., [10, 1]) that the MG for the discrete Poisson problem has a fast convergence rate. The latter suggests that the nonsymmetric perturbation is well controlled by the Laplace operator whose pseudodifferential order is 2, and, hence, MG can be applied efficiently. In particular, the second approach adopted in this paper is the V-cycle MG that uses the exact coarsest grid solver and in which smoothing operators depend only on the discrete Laplace operator. Application of the convergence theory in [6] guarantees that the MG $V(1, 0)$ cycle and, hence, the $V(\nu_1, \nu_2)$ cycles for $\nu_1 \geq 1$ and $\nu_2 \geq 0$ converge uniformly in the finest mesh size and the number of refinement levels, provided that the coarsest mesh size is sufficiently small. This condition indicates that the direct MG is probably expensive because it requires us to solve a relatively large coarse grid problem. The coarsest mesh size used in our experiments for both approaches is $1/4$. Hence, it is clear that such a condition is not essential for our problem.

The present paper attempts to confirm theories numerically on the accuracy of finite element approximation and the performance of solution methods. For the Dirichlet problem on the L-shape domain, we show here that finite element approximations using continuous piecewise linear elements converge to the exact solution with $O(h^2)$ errors in the *discrete* H^1 seminorm and L^2 norm, respectively. The theoretically predicted error bounds are $O(h)$ in the H^1 norm and $O(h^{\frac{5}{3}})$ in the L^2 norm. We appear to have obtained superconvergence in the discrete H^1 seminorm for this particular case. We seem to achieve the optimal L^2 accuracy, which is better than what is predicted by our theory. We show that the SM formula using the MG V-cycles and the direct MG V-cycle converge independently of the finest mesh size h and the number of refinement levels. We also show that the full MG computes a final approximation with accuracy on the order of a discretization error in a total amount of work equal to about nine relaxation sweeps on the finest grid.

The paper is organized as follows. The finite element method and related results from [5] are introduced in sections 2 and 3, and the solution methods are described in section 4. We present the results of numerical experiments in section 5. Some conclusions and final remarks are contained in section 6.

2. The problem and preliminaries. As a model problem, we consider the Poisson equation with homogeneous Dirichlet boundary conditions

$$\begin{cases} -\Delta u &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where f is a given function in $L^2(\Omega)$ and Ω is an open, bounded polygonal domain in $\mathcal{R}^2$ with one re-entrant angle. Extension to the domain with the finite number of re-entrant angles is straightforward. We use the standard notation and definition for the Sobolev spaces $H^t(B)$ for $t \geq 0$; the standard associated inner products are denoted by $(\cdot, \cdot)_{t,B}$, and their respective norms and seminorms are denoted by $\|\cdot\|_{t,B}$ and $|\cdot|_{t,B}$. The space $L^2(B)$ is interpreted as $H^0(B)$, in which case the inner product and norm will be denoted by $(\cdot, \cdot)_B$ and $\|\cdot\|_B$, respectively.

Let ω be the internal angle of Ω satisfying $\pi < \omega < 2\pi$. Without loss of generality, assume that the corresponding vertex is at the origin. Define the *singular* and the

dual singular functions by

$$s = r^{\frac{\pi}{\omega}} \sin\frac{\pi\theta}{\omega} \quad \text{and} \quad s_- = r^{-\frac{\pi}{\omega}} \sin\frac{\pi\theta}{\omega}, \tag{2.2}$$

respectively, in the polar coordinates (r, θ). The coordinates are chosen at the origin so that the internal angle ω is spanned by the two half-lines $\theta = 0$ and $\theta = \omega$. Set

$$B(r_1; r_2) = \{(r, \theta) : \ r_1 < r < r_2 \text{ and } 0 < \theta < \omega\} \cap \Omega \quad \text{and} \quad B(r_1) = B(0; r_1).$$

Define a family of cut-off functions of r, $\eta_\rho(r)$ as follows:

$$\eta_\rho(r) = \begin{cases} 1 & \text{in } B(\frac{\rho R}{2}), \\ \frac{15}{16}\Big\{\frac{8}{15} - \big(\frac{4r}{\rho R} - 3\big) + \frac{2}{3}\big(\frac{4r}{\rho R} - 3\big)^3 - \frac{1}{5}\big(\frac{4r}{\rho R} - 3\big)^5\Big\} & \text{in } \bar{B}(\frac{\rho R}{2}; \rho R), \\ 0 & \text{in } \Omega \setminus \bar{B}(\rho R), \end{cases} \tag{2.3}$$

where ρ is a parameter in (0, 2] and $R \in \mathcal{R}$ is a fixed number so that the $\eta_2 s$ vanishes identically on $\partial\Omega$. It is well known (see, e.g., [8, 9]) that the solution of (2.1) has the singular function representation

$$u = w + \lambda\eta_\rho s, \tag{2.4}$$

where $w \in H^2(\Omega) \cap H_0^1(\Omega)$ is the regular part of the solution and $\lambda \in \mathcal{R}$ is the so-called *stress intensity factor*.

In [5], by using a new *extraction formula* in terms of w,

$$\lambda = \frac{1}{\pi}(w, \Delta(\eta_2 s_-))_{B(R;2R)} + \frac{1}{\pi}(f, \eta_2 s_-)_{B(2R)}, \tag{2.5}$$

we are able to deduce a well-posed integro-differential equation of w,

$$-\Delta w - \frac{1}{\pi}(w, \Delta(\eta_2 s_-))_{B(R;2R)}\Delta(\eta_\rho s) = f + \frac{1}{\pi}(f, \eta_2 s_-)_{B(2R)}\Delta(\eta_\rho s) \quad \text{in } \Omega, \tag{2.6}$$

where $0 < \rho \le 1$. Its variational formulation is to find $w \in H_0^1(\Omega)$ such that

$$a(w, v) = g(v) \quad \forall\ v \in H_0^1(\Omega), \tag{2.7}$$

where the bilinear form $a(\cdot, \cdot)$ and linear form $g(\cdot)$ are defined by

$$a(w, v) = a^s(w, v) + b(w, v), \quad b(w, v) = -\frac{1}{\pi}(w, \Delta(\eta_2 s_-))_{B(R;2R)}(\Delta(\eta_\rho s), v)_{B(\frac{\rho R}{2};\rho R)},$$
$$a^s(w, v) = (\nabla w, \nabla v)_\Omega, \quad \text{and} \quad g(v) = (f, v)_\Omega + \tfrac{1}{\pi}(f, \eta_2 s_-)_{B(2R)}(\Delta(\eta_\rho s), v)_{B(\frac{\rho R}{2};\rho R)}.$$

Note that the bilinear form $b(\cdot, \cdot)$ is not symmetric. It was shown in [5] that (2.7) has a unique solution $w \in H_0^1(\Omega) \cap H^2(\Omega)$. The following continuity and coercivity bounds were also established:

$$|a^s(w, v)| \le C\,|w|_{1,\Omega}\,|v|_{1,\Omega}, \quad |b(w, v)| \le C\,\|w\|_\Omega\,\|v\|_\Omega \quad \forall\ w,\ v \in H_0^1(\Omega), \tag{2.8}$$

$$\text{and} \quad |v|_{1,\Omega}^2 - C\|v\|_\Omega^2 \le a(v, v) \quad \forall\ v \in H_0^1(\Omega). \tag{2.9}$$

Here and henceforth, we use C with or without script to denote a generic positive constant independent of the mesh size h and the number of levels J introduced in section 4.

3. Finite element approximation. Let $\mathcal{T}_h$ be a partition of the domain Ω into triangular finite elements; i.e., $\Omega = \cup_{K\in\mathcal{T}_h} K$ with $h = \max\{\operatorname{diam} K\colon\ K \in \mathcal{T}_h\}$. Assume that the triangulation $\mathcal{T}_h$ is regular. Set

$$\mathcal{V}_h = \{\phi_h \in C^0(\Omega)\colon\ \phi_h|_K \text{ is linear, } \forall\ K \in \mathcal{T}_h,\ \phi_h = 0 \text{ on } \partial\Omega\} \subset H_0^1(\Omega).$$

Then the finite element approximation to (2.7) in $H_0^1(\Omega)\cap H^2(\Omega)$ becomes as follows: find $w_h \in \mathcal{V}_h$ such that

$$a(w_h, v) = g(v) \quad \forall\ v \in \mathcal{V}_h. \tag{3.1}$$

Approximations to the stress intensity factor and the solution of (2.1) can be computed according to (2.5) and (2.4) as follows:

$$\lambda_h = \frac{1}{\pi}(w_h, \Delta(\eta_2 s_-))_{B(R;2R)} + \frac{1}{\pi}(f, \eta_2 s_-)_{B(2R)} \tag{3.2}$$

$$\text{and} \quad u_h = w_h + \lambda_h \eta_\rho s. \tag{3.3}$$

The following error estimates were established in [5].

THEOREM 3.1. *Assume that $w \in H^2(\Omega)$ and u are the solutions of* (2.7) *and* (2.1), *respectively. Then the following error estimates hold:*

$$\|w - w_h\|_{1,\Omega},\ \|u - u_h\|_{1,\Omega} \le C\, h\|f\|_\Omega, \tag{3.4}$$

$$\textit{and} \quad \|w - w_h\|_\Omega,\ \|u - u_h\|_\Omega,\ |\lambda - \lambda_h| \le C\, h^{1+\frac{\pi}{\omega}}\|f\|_\Omega. \tag{3.5}$$

4. Solution methods. Let $\{\phi_j(x)\}_{j=1}^N$ be nodal bases for $\mathcal{V}_h$, write $w_h = \sum_{j=1}^N w_j^h \phi_j(x)$, and then the matricial form of (3.1) is

$$\sum_{j=1}^{N} w_j^h a(\phi_j,\, \phi_i) = g(\phi_i) \tag{4.1}$$

for $i = 1, \dots, N$. Let A_h^s and B_h be $N \times N$ matrices with $(i,\, j)$ entries $a^s(\phi_j,\, \phi_i)$ and $b(\phi_j,\, \phi_i)$, respectively, let W_h be the unknown vector with components w_i^h, and let F_h be the right-hand side vector with components $g(\phi_i)$. Then (4.1) may be written as

$$A_h W_h = F_h, \tag{4.2}$$

where $A_h = A_h^s + B_h$. Note that B_h is a nonsymmetric and rank-one matrix. In this section, we discuss two approaches for solving the system of linear equations in (4.2). The first approach is to use the SM formula which requires two inversions of the discrete Laplace operator A_h^s. The second approach is the application of the MG V-cycle with smoothers depending only on A_h^s.

4.1. The SM formula. Any rank-one matrix B_h can be written as the product of the column and row vectors. In particular, we have

$$B_h = -UV^t,$$

where U and V are column vectors with ith components

$$U_i = \frac{1}{\pi}(\Delta(\eta_\rho s), \phi_i)_{B(\frac{\rho R}{2};\rho R)} \quad \text{and} \quad V_i = (\phi_i,\, \Delta(\eta_2 s_-))_{B(R;\,2R)},$$

respectively. By the SM formula, the inverse of A_h has the form

$$A_h^{-1} = (A_h^s)^{-1} + \alpha\,(A_h^s)^{-1}UV^t(A_h^s)^{-1}, \quad \text{where} \quad \alpha = \frac{1}{1 - V^t(A_h^s)^{-1}U}.$$

Therefore, the solution of the system of linear equations in (4.2), $W_h = A_h^{-1}F_h$, can be computed by the following algorithm.

1. Solve $A_h^s X = F_h$ for X.
2. Solve $A_h^s Y = U$ for Y.
3. Compute $\alpha = \frac{1}{1-V^tY}$ and $\beta = V^tX$.
4. Set $W_h = X + \alpha\,\beta\,Y$.

Note that the above algorithm requires *two* (approximate) inversions of the discrete Poisson equations, which will be done by using MG. It is well known that MG applied to the discrete Poisson equation has a fast convergence rate.

4.2. MG algorithms. In this subsection, we describe the MG for solving the discrete problem in (4.2); the MG uses the exact coarsest grid solver, and smoothing operators on the fine grids depend only on A_h^s. To this end, we start with an intentionally coarse triangulation $\mathcal{T}_0$ of $\bar{\Omega}$ with the properties that the boundary $\partial\Omega$ is composed of edges of some triangles K in $\mathcal{T}_0$ and that every triangle of $\mathcal{T}_0$ is shape regular. Each triangle K of $\mathcal{T}_0$ is regularly refined several times, giving a family of nested triangulations $\mathcal{T}_0, \mathcal{T}_1, \ldots, \mathcal{T}_J \equiv \mathcal{T}$, such that each triangle of $\mathcal{T}_{k+1}$ is generated by subdividing a triangle of $\mathcal{T}_k$ into four congruent triangles. Let h_k be the mesh size for the corresponding triangulation $\mathcal{T}_k$; we then have $h_k = 2h_{k+1}$. Denote by h the mesh size of the finest grid. We associate the triangulation $\mathcal{T}_k$ with the continuous piecewise linear finite element space $\mathcal{V}_k$. It is easy to see that the family of spaces $\{\mathcal{V}_k\}$ is nested.

For $k = 0, 1, \ldots, J$, define the nonsymmetric and symmetric elliptic operators $\mathcal{A}_k$, $\mathcal{A}_k^s : \mathcal{V}_k \longrightarrow \mathcal{V}_k$ by

$$(\mathcal{A}_k w, \phi) = a(w, \phi) \quad \forall\, \phi \in \mathcal{V}_k \quad \text{and} \quad (\mathcal{A}_k^s w, \phi) = a^s(w, \phi) \quad \forall\, \phi \in \mathcal{V}_k,$$

respectively. Also, define the L^2-projection operator $\mathcal{P}_k : \mathcal{V}_{k+1} \longrightarrow \mathcal{V}_k$ by

$$(\mathcal{P}_k w, \phi) = (w, \phi) \quad \forall\, \phi \in \mathcal{V}_k.$$

The smoothing operators $\mathcal{R}_k : \mathcal{V}_k \longrightarrow \mathcal{V}_k$ adopted in this paper are the exact solver on the coarsest grid and the Gauss–Seidel iteration based on the symmetric operator $\mathcal{A}_k^s$ for $k = 1, 2, \ldots, J$. Now we define the MG operator $\mathcal{B}_k : \mathcal{V}_k \longrightarrow \mathcal{V}_k$ of the V(ν_1, ν_2)-cycle by induction (see [1]).

V(ν_1, ν_2)-Algorithm. *Set $\mathcal{B}_0 = \mathcal{A}_0^{-1}$. Assume that $\mathcal{B}_{k-1}$ has been defined, and define $\mathcal{B}_k\, g$ for $g \in \mathcal{V}_k$ as follows.*

1. *Set an initial guess x_0.*
2. *Define x^ℓ for $\ell = 1, \ldots, \nu_1$ by $x^\ell = x^{\ell-1} + \mathcal{R}_k(g - \mathcal{A}_k x^{\ell-1})$.*
3. *Define x^{ν_1+1} by $x^{\nu_1+1} = \mathcal{B}_{k-1}\mathcal{P}_{k-1}(g - \mathcal{A}_k x^{\nu_1})$.*
4. *Set $\mathcal{B}_k g = x^{\nu_1+\nu_2+1}$, where x^ℓ is defined for $\ell = \nu_1 + 2, \ldots, \nu_1 + \nu_2 + 1$ by $x^\ell = x^{\ell-1} + \mathcal{R}_k(g - \mathcal{A}_k x^{\ell-1})$.*

It was shown in [6] that if the bilinear forms $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ satisfy the continuity and coercivity bounds in (2.8) and (2.9), then the MG $V(1,0)$-cycle and, hence, the $V(\nu_1, \nu_2)$-cycles for $\nu_1 \geq 1$ and $\nu_2 \geq 0$ converge independently of the mesh size h and the number of levels J, provided that the coarsest mesh size h_0 is sufficiently small. This condition on h_0 indicates that this direct MG approach is probably expensive

because it requires us to solve a relatively large coarse grid problem. Numerical experiments presented in the next section use $h_0 = 1/4$. Hence, it is clear that such a condition is not essential for our problem.

Finally, we will also consider the full MG scheme FMG(nV(ν_1, ν_2)), which proceeds from the coarsest to the finest grid, invoking n $V(\nu_1, \nu_2)$ cycles on each level along the way with $n \geq 1$ (see [4]). Thus, each coarse level serves to provide a good initial approximation to the next finer level, with the intent of producing a final approximation on the finest grid that is accurate to the level of discretization error.

5. Numerical experiments. In this section, we present the results of numerical experiments. Performance of the solution methods and the finite element discretization accuracy are first analyzed individually, and then the overall FMG performance is assessed.

We consider the Poisson equation in (2.1) over the L-shaped domain

$$\Omega = ((-1,\, 1) \times (-1,\, 1)) \setminus ([0,\, 1) \times (-1,\, 0])\,.$$

The domain Ω is first partitioned into $3(n \times n)$ square subdomains of side length $h = 1/n$. The $3(n \times n)$ subsquares are then divided into pairs of triangles by connecting the bottom right and upper left corners. We use continuous piecewise linear finite element space for the approximation of the regular part w of the solution with respect to the triangulation with the grid interval h ranging from 2^{-2} to 2^{-7} for each direction. In the performances of the MG V-cycles and the full MG V-cycles, the mesh size of the coarsest grid is chosen to be $h_0 = 2^{-2}$. To increase the accuracy of numerical quadratures, all integrations, involving singular and dual singular functions, on a given triangle $K \in \mathcal{T}_k$ are computed by the composite three points quadrature rule using small triangles of the side length 2^{-11}. These small triangles are generated by $(9 - k)$-times subdividing the triangles K of the side length $2^{-(k+2)}$. We will use the following discrete L^2-norm and H^1-seminorm for the vector $V_h \in \Re^N$:

$$\|V_h\|_h := \sqrt{\left(M_h V_h, V_h\right)} \quad \text{and} \quad |V_h|_{1,h} := \sqrt{\left(A_J^s V_h, V_h\right)},$$

where M_h is the mass matrix associating with $\mathcal{V}_h$ and A_J^s is the stiffness matrix of the Laplace operator defined in section 4.

5.1. Performance of the solution methods. To study the performance of the solution methods, we begin with the Poisson equation in (2.1) with the homogeneous right-hand side; i.e.,

$$\begin{cases} -\Delta u &= 0 \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{cases}$$

It is obvious that the exact solution is zero. With the initial guess of one, we report asymptotic convergence factors for the MG V-cycle defined in section 4.2 (referring to the direct MG) and the iterative Sherman–Morrison algorithm (SMA). Let X^k and Y^k denote, respectively, the kth V(ν_1,ν_2) iterates of the discrete Poisson problems in the first two steps of the SMA defined in section 4.1. The iterative SMA is defined as follows:

$$W_h^k = X^k + \alpha^k \beta^k Y^k$$

TABLE 1
Convergence factors for direct MG $V(\nu_1, \nu_2)$ cycle.

Mesh size	$\rho = 1.00$			$\rho = 0.50$		
	V(1,0)	V(2,0)	V(1,1)	V(1,0)	V(2,0)	V(1,1)
$h = \frac{1}{8}$	0.3971	0.1822	0.1830	0.3988	0.1720	0.1694
$h = \frac{1}{16}$	0.4415	0.2460	0.2369	0.4564	0.2793	0.2758
$h = \frac{1}{32}$	0.4417	0.2696	0.2524	0.4504	0.2574	0.2866
$h = \frac{1}{64}$	0.4642	0.3113	0.2924	0.4360	0.2691	0.2598
$h = \frac{1}{128}$	0.4719	0.3081	0.2612	0.4777	0.3242	0.2831

TABLE 2
Convergence factors for the iterative SMA with $\rho = 1.0$.

Mesh size	V(1,0)	V(2,0)	V(1,1)
$h = \frac{1}{8}$	0.4005	0.1709	0.1691
$h = \frac{1}{16}$	0.4588	0.2722	0.2780
$h = \frac{1}{32}$	0.4563	0.2421	0.2680
$h = \frac{1}{64}$	0.4638	0.2285	0.2553
$h = \frac{1}{128}$	0.4369	0.3072	0.2474

with $\alpha^k = \frac{1}{1-V^tY^k}$ and $\beta^k = V^tX^k$. Convergence factors σ_h^k here are defined as ratios of a successive discrete H^1-seminorm of errors,

$$\sigma_h^k := \frac{|e_h^{k+1}|_{1,h}}{|e_h^k|_{1,h}} \qquad \text{with} \quad e_h^k = W_h - W_h^k,$$

where $W_h = 0$ and W_h^k are the exact solution and the kth iterate of (4.2) with $f = 0$, respectively. It is clear that the convergence factor of the iterative SMA does not depend on ρ and the mesh size h since the MG iterations apply to two discrete Poisson problems.

Tables 1 and 2 represent convergence factors measured after 20 MG V(ν_1, ν_2) cycles for the direct MG and the iterative SMA, respectively. Uniform convergence theory of the direct MG (see the discussion in section 4.2) requires the condition that h_0 is sufficiently small. Clearly, the numerical results reported here show that convergence of the direct MG is not subject to such a constraint since we used $h_0 = 1/4$. The observed convergence factors in Tables 1 and 2 for the direct MG and the iterative SMA are almost same, but the latter costs about twice as many arithmetic operations as the former for one re-entrant corner and much more for several re-entrant corners. This indicates that the direct MG is the method of choice. One V$(2, 0)$ or one V$(1, 1)$ cycle costs about twice as much as one V$(1, 0)$ cycle. Table 1 shows that the reduction rate by two V$(1, 0)$ cycles is better than the reduction rate of one V$(2, 0)$ or V$(1, 1)$ cycle. This suggests that the MG V$(1, 0)$ is the most effective solver, which is confirmed by the subsequent tables.

5.2. Discretization errors and FMG performance. To measure the discretization error, we consider the Poisson equation in (2.1) with a known nonzero

TABLE 3
The number of iterations for the direct MG till $|W_h - W_h^k|_{1,h} < 10^{-4}$.

Mesh size	Dim.	$\rho = 1.00$			$\rho = 0.50$		
		V(1,0)	V(2,0)	V(1,1)	V(1,0)	V(2,0)	V(1,1)
$h = \frac{1}{4}$	33	1	1	1	1	1	1
$h = \frac{1}{8}$	161	10	7	6	10	7	6
$h = \frac{1}{16}$	705	11	8	7	11	7	7
$h = \frac{1}{32}$	2945	12	8	7	11	8	7
$h = \frac{1}{64}$	12033	12	8	7	11	8	7
$h = \frac{1}{128}$	48641	12	8	7	12	8	7

TABLE 4
The number of iterations till $|W_h - W_h^k|_1 < 10^{-4}$ *for* $\rho = 1.0$ *(the iterative SMA).*

Mesh Size	Dim.	V(1,0)	V(2,0)	V(1,1)
$h = \frac{1}{4}$	33	1	1	1
$h = \frac{1}{8}$	161	10	7	6
$h = \frac{1}{16}$	705	11	7	7
$h = \frac{1}{32}$	2945	11	8	7
$h = \frac{1}{64}$	12033	11	8	7
$h = \frac{1}{128}$	48641	11	8	7

solution by choosing the right-hand side function to be

$$f = \begin{cases} \sin(2\pi x)\big[2\pi^2(y^2+2y)(y^2-1) - (6y^2+6y-1)\big] - \Delta(\eta_\rho s) & \text{if } -1 \le y < 0, \\ \sin(2\pi x)\big[2\pi^2(-y^2+2y)(y^2-1) - (-6y^2+6y+1)\big] - \Delta(\eta_\rho s) & \text{if } 0 < y \le 1, \end{cases}$$

where $s = r^{\frac{2}{3}} \sin(\frac{2\theta}{3})$ is the singular function. The exact solution of the underlying problem is then

$$u = w + \eta_\rho s,$$

where w is the regular part of the solution, which is the exact solution of (2.7) given by

$$w = \begin{cases} \sin(2\pi x)(\frac{1}{2}y^2 + y)(y^2-1) & \text{if } -1 \le y \le 0, \\ \sin(2\pi x)(-\frac{1}{2}y^2 + y)(y^2-1) & \text{if } 0 \le y \le 1. \end{cases}$$

Note that the function w is in $H^2(\Omega)$ but not in $H^3(\Omega)$. Let W_h be the exact solution of (4.2), the coefficient vector of the finite element approximation. We first depict the number of iterations required in order to reduce the initial error $|W_h - W_h^0|_{1,h} = |W_h|_{1,h}$ within the given tolerance $\varepsilon = 10^{-4}$ in Tables 3 and 4.

Next, we compute the approximate solution $\hat{W}_h$ of (4.2) on each of the various levels of discretization by using 30 direct MG $V(1,0)$ cycles. This is to ensure that the errors reported in Tables 5 and 6 properly reflect discretization accuracy without contamination from algebraic iteration errors. Let w_h^I be the interpolant of w in $\mathcal{V}_h = \text{span}\{\phi_i(x)\}_{i=1}^N$; i.e.,

$$w_h^I = \sum_{i=1}^{N} w_j^I \phi_i(x),$$

TABLE 5
The discrete L^2-norm errors and the convergence rates for w.

	$\rho = 1.00$		$\rho = 0.50$	
Mesh size	L^2-NORM	RATE	L^2-NORM	RATE
$h = \frac{1}{4}$	2.5729e-02		2.2269e-02	
$h = \frac{1}{8}$	7.5903e-03	1.7612	6.6963e-03	1.7336
$h = \frac{1}{16}$	2.0062e-03	1.9197	1.7669e-03	1.9221
$h = \frac{1}{32}$	5.0926e-04	1.9780	4.4820e-04	1.9790
$h = \frac{1}{64}$	1.2774e-04	1.9952	1.1238e-04	1.9958
$h = \frac{1}{128}$	3.1850e-05	2.0038	2.8019e-05	2.0039

TABLE 6
The discrete H^1 seminorm and the convergence rates for w.

	$\rho = 1.00$		$\rho = 0.50$	
Mesh size	H^1-NORM	RATE	H^1-NORM	RATE
$h = \frac{1}{4}$	1.6457e-01		1.6027e-01	
$h = \frac{1}{8}$	4.7468e-02	1.7937	4.5217e-02	1.8256
$h = \frac{1}{16}$	1.2568e-02	1.9172	1.1800e-02	1.9381
$h = \frac{1}{32}$	3.1938e-03	1.9764	3.0068e-03	1.9725
$h = \frac{1}{64}$	8.0209e-04	1.9934	7.5437e-04	1.9949
$h = \frac{1}{128}$	2.0121e-04	1.9951	1.8791e-04	2.0052

where w_i^I is the nodal value of w. Let W_I be the coefficient vector of w_h^I, i.e., the vector with components w_i^I. Define the error vector by

$$E(h) = W_I - \hat{W}_h.$$

The rates of convergence for discretization errors in the discrete L^2 norm, the discrete H^1 seminorm, and the absolute value are measured by

$$\log_2 \frac{\|E(h)\|_h}{\|E(\frac{h}{2})\|_{\frac{h}{2}}}, \quad \log_2 \frac{\|E(h)\|_{1,h}}{\|E(\frac{h}{2})\|_{1,\frac{h}{2}}}, \quad \text{and} \quad \log_2 \frac{|\lambda - \lambda_h|}{|\lambda - \lambda_{\frac{h}{2}}|},$$

respectively.

Numerical results given in Tables 5 and 6 show that the discretization accuracy of the finite element approximation to w is $O(h^2)$ with respect to both the discrete L^2 norm and the discrete H^1 seminorm for $\rho = 1.0$ and 0.5. The theoretically predicted error bounds are only $O(h^{\frac{5}{3}})$ in the L^2 norm and $O(h)$ in the H^1 norm. We therefore appear to have obtained optimal convergence in the L^2 and superconvergence in the discrete H^1 seminorm for this particular case. Our finite element theory in [5] requires that the mesh size h is sufficiently small since we used Gårding's inequality in its analysis. It is clear from Tables 5 and 6 that our finite element approximation is not subject to this constraint.

To test overall accuracy for the FMG, we studied the FMG based on $5\,V(1,0)$, $4\,V(2,0)$, and $4\,V(1,1)$ cycles. Tables 7 and 8 show that the total errors produced by the FMG are comparable to those discretization errors estimated in the respective Tables 5 and 6. This indicates that the underlying problem on level h can be solved to within the level of discretization error at a cost of about $\frac{4}{3} \cdot 5V(1,0) \approx 7\,V(1,0)$ or

TABLE 7
The discrete L^2-norm errors and the convergence rates by FMG.

	FMG(5V(1,0))		FMG(4V(2,0))		FMG(4V(1,1))	
Mesh size	ERROR	RATE	ERROR	RATE	ERROR	RATE
$h = \frac{1}{8}$	7.6617e-03	1.7477	7.6010e-03	1.7591	7.6114e-03	1.7572
$h = \frac{1}{16}$	2.0498e-03	1.9022	2.0296e-03	1.9050	2.0283e-03	1.9079
$h = \frac{1}{32}$	5.2220e-04	1.9728	5.1803e-04	1.9701	5.1511e-04	1.9773
$h = \frac{1}{64}$	1.3105e-04	1.9945	1.3012e-04	1.9932	1.2912e-04	1.9962
$h = \frac{1}{128}$	3.2667e-05	2.0042	3.2446e-05	2.0037	3.2179e-05	2.0045

TABLE 8
The discrete H^1-seminorm errors and the convergence rates by FMG.

	FMG(5V(1,0))		FMG(4V(2,0))		FMG(4V(1,1))	
Mesh size	ERROR	RATE	ERROR	RATE	ERROR	RATE
$h = \frac{1}{8}$	4.7954e-02	1.7790	4.7389e-02	1.7961	4.7465e-02	1.7938
$h = \frac{1}{16}$	1.2973e-02	1.8861	1.2672e-02	1.9029	1.2669e-02	1.9056
$h = \frac{1}{32}$	3.3488e-03	1.9538	3.2528e-03	1.9619	3.2237e-03	1.9745
$h = \frac{1}{64}$	8.6492e-04	1.9530	8.2053e-04	1.9871	8.0915e-04	1.9942
$h = \frac{1}{128}$	2.3739e-04	1.8653	2.0634e-04	1.9915	2.0290e-04	1.9956

TABLE 9
The absolute value errors and the convergence rates for λ.

	$\rho = 1.00$		$\rho = 0.50$	
Mesh size	$\lvert\lambda - \lambda_h\rvert$	RATE	$\lvert\lambda - \lambda_h\rvert$	RATE
$h = \frac{1}{4}$	7.0471e-02		6.8240e-02	
$h = \frac{1}{8}$	2.0457e-02	1.7844	2.0362e-02	1.7447
$h = \frac{1}{16}$	5.3595e-03	1.9324	5.3451e-03	1.9296
$h = \frac{1}{32}$	1.3576e-03	1.9810	1.3449e-03	1.9907
$h = \frac{1}{64}$	3.4091e-04	1.9936	3.2827e-04	2.0345
$h = \frac{1}{128}$	8.5640e-05	1.9930	7.3006e-05	2.1688

TABLE 10
The absolute value errors and the convergence rates by FMG.

	FMG(5V(1,0))		FMG(4V(2,0))		FMG(4V(1,1))	
Mesh size	ERROR	RATE	ERROR	RATE	ERROR	RATE
$h = \frac{1}{8}$	2.0710e-02	1.7667	2.0430e-02	1.7863	2.0454e-02	1.7846
$h = \frac{1}{16}$	5.4565e-03	1.9243	5.3802e-03	1.9250	5.3889e-03	1.9243
$h = \frac{1}{32}$	1.3846e-03	1.9785	1.3672e-03	1.9764	1.3650e-03	1.9811
$h = \frac{1}{64}$	3.4781e-04	1.9931	3.4360e-04	1.9924	3.4259e-04	1.9943
$h = \frac{1}{128}$	8.7360e-05	1.9933	8.6318e-05	1.9930	8.6045e-05	1.9933

$5\,V(2,0)$ or $5\,V(1,1)$ cycles on level h. Finally, results for the stress intensity factor are contained in Tables 9 and 10.

6. Conclusion and remarks. We study the MG V-cycle applied directly to the discrete problem in (4.2) and used in the SMA. Our numerical study shows that

the direct MG is much more efficient than the iterative SMA. Our numerical study confirms theories on finite element accuracy established in [5] and MG convergence discussed in section 4.2. It especially shows that our finite element method seems to be optimally accurate in the L^2 norm. The theory for MG convergence is subject to the constraint that the coarsest mesh size is sufficiently small. But this condition is not essential numerically since we used $h_0 = 1/4$ in our experiments.

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