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A FINITE ELEMENT METHOD USING SINGULAR FUNCTIONS FOR THE POISSON EQUATION: CORNER SINGULARITIES*

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Abstract. Consider the Poisson equation with homogeneous Dirichlet boundary conditions on a polygonal domain with one reentrant corner. In this paper, we develop a new finite element method for the accurate computation of the solution and stress intensity factors. It is well known that the solution of such a problem has a singular function representation: $u = w + \lambda\eta s$, where $w \in H^2(\Omega) \cap H_0^1(\Omega)$, $\lambda \in \mathcal{R}$ and η are the stress intensity factor and cut-off function, respectively, and s is a known singular function depending only on the reentrant angle. By using the dual singular and an extra cut-off function, we are able to derive a new extraction formula for λ in terms of w and, hence, deduce a well-posed variational problem for w. Standard continuous piecewise linear finite element approximation yields $O(h)$ optimal accuracy for w, which, in turn, implies the same accuracy for u in the H^1 norm. We are able only to prove $O(h^{1+\frac{\pi}{\omega}})$ error bounds for w and u in the L^2 norm and for λ in the absolute value, where ω is the internal angle.

Key words. corner singularity, finite element, stress intensity factor

AMS subject classifications. 65F10, 65F30

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1. Introduction.

Solutions of elliptic boundary value problems on a domain with corners have singular behavior near the corners. This occurs even when data of the underlying problem are very smooth. Such singular behavior affects the accuracy of the finite element method throughout the whole domain. As a model problem, we consider the Poisson equations with homogeneous Dirichlet boundary conditions:

$$\begin{cases} -\Delta u = f & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1.1}$$

where Δ stands for the Laplacian operator, f is a given function in $L^2(\Omega)$, and Ω is an open, bounded polygonal domain in $\mathcal{R}^2$ with one reentrant corner. Extension to the domain with a finite number of reentrant corners is straightforward.

Let ω be the internal angle of Ω satisfying $\pi < \omega < 2\pi$. Without the loss of generality, assume that the corresponding vertex is at the origin. It is well known (cf. [3, 11, 16]) that there exists a unique solution $u \in H_0^1(\Omega)$ of (1.1) and, in addition, there exists a unique number $\lambda \in \mathcal{R}$, the so-called *stress intensity factor*, such that

$$u - \lambda s \in H^2(\Omega). \tag{1.2}$$

Here, the *singular function* s is defined as

$$s = r^{\frac{\pi}{\omega}} \sin \frac{\pi\theta}{\omega} \tag{1.3}$$

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in the polar coordinates (r, θ) which are chosen at the origin so that the internal angle ω is spanned by the two half-lines $\theta = 0$ and $\theta = \omega$. Let η be a smooth cut-off function which equals one identically in a neighborhood of the origin, and the support of η is small enough so that the function ηs vanishes identically on $\partial\Omega$. From (1.2), the solution of problem (1.1) has the representation

$$u = w + \lambda\eta s, \tag{1.4}$$

where $w \in H^2(\Omega) \cap H_0^1(\Omega)$ satisfies

$$-\Delta w - \lambda\Delta(\eta s) = f \quad \text{in } \Omega. \tag{1.5}$$

Moreover, the following regularity estimate holds:

$$\|w\|_2 + |\lambda| \le C_R \|f\|, \tag{1.6}$$

where C_R is a positive constant depending on the domain and the diameter of the support of η. C_R especially increases if the diameter of η is chosen smaller. Define the *dual singular function* as

$$s_- = r^{-\frac{\pi}{\omega}} \sin \frac{\pi\theta}{\omega}. \tag{1.7}$$

Note that both s and s_- are harmonic functions in Ω. The stress intensity factor can be expressed in terms of u by the following *extraction formula* (see [16]):

$$\lambda = \frac{1}{\pi}\left(\int_\Omega f\eta s_-\, dx + \int_\Omega u\Delta(\eta s_-)\, dx\right). \tag{1.8}$$

It is well known (cf. [16]) that the solution, u, of problem (1.1) is in $H^r(\Omega)$ for $r < 1 + \frac{\pi}{\omega}$. Such lack of regularity affects the accuracy of the finite element approximation and, hence, the approximation to the stress intensity factor. There were several approaches in the literatures for overcoming this difficulty. One is based on local mesh refinement (see, e.g., [1, 17, 18, 19, 20]). The advantage of the method of local mesh refinement is that the knowledge of the exact forms of the singular functions is not needed. Another is done by augmenting the space of trial functions in which one looks for the approximate solution (see, e.g., [14, 12, 5, 6, 8, 13]). Some other approaches may be found in [2, 21].

Note that the regular part of the solution, w, and the stress intensity factor, λ, are related through (1.4), (1.5), and (1.8). Based on such observation, the dual singular function method (cf. [13, 5, 6]) was implemented as an iterative procedure which iterates back and forth between these equations. Recently, this approach was extended to full multigrid versions (cf. [7, 8, 9]).

The purpose of this paper is to develop and analyze a new finite element method for the accurate computation of the solution and intensity factors. Note that the loss of the standard finite element approximation's accuracy is due to the lack of the solution's smoothness and that the regular part of the solution ($w \in H^2(\Omega)$) is smoother than the solution itself ($u \in H^r(\Omega)$ with $r < 1+\frac{\pi}{\omega}$). Therefore, it is natural to first approximate w and then compute λ and u. To do so, we decouple the system of (1.4), (1.5), and (1.8) by using the dual singular function and another cut-off function with a support bigger than that of η. Now the w is uniquely determined by a well-posed variational problem, the λ can be expressed by an extraction formula in terms of w, and the u can be computed by formula (1.4). Based on this variational problem, continuous piecewise linear finite element approximation yields $O(h)$ optimal accuracy for w, which, in turn, implies the same accuracy for u in the H^1 norm. We use the standard duality argument to establish the error bound for w in the L^2 norm. In order to obtain $O(h^2)$ optimal accuracy, it requires the H^2 regularity estimate for the adjoint problem. Even though the variational problem for w is H^2 regular (see (1.6)), we are unable to show that its adjoint is H^2 regular at this stage. Instead, we make use of an adjoint problem with a simplified linear form that has only H^r $(r < 1 + \frac{\pi}{\omega})$ regularity. Therefore, the error bound in the L^2 norm for w that we can prove here is $O(h^{1+\frac{\pi}{\omega}})$ and, hence, the same error bounds for λ in the absolute value and for u in the L^2 norm. For numerical experiments, see [10].

The system of linear equations arising from the finite element discretization for w is nonsymmetric. This is due to the correction term in the w equation (see (2.8)). Note that such a correction term corresponds to a rank-one integral operator. Use of the rank-one property of this operator leads to the Sherman–Morrison formula for solving the algebraic equation, which requires two (approximate) inversions of the discrete Laplacian operator. The fact that the integral operator is well-controlled by the Laplacian operator indicates that V-cycle multigrid algorithms with smoothing operators, which depend only on the discrete Laplacian operator, and the exact coarsest grid solver are efficient. For more details, see [10].

The paper is organized as follows. The variational form for w and extraction formula for λ are developed in section 2. We establish the well-posedness of the variational form in section 3; the finite element method and its error analysis are given in section 4; proofs of two lemmas involving lengthy computations are presented in section 5.

2. A new formulation. In this section, we introduce an extraction formula in terms of w and a new variational formulation which uniquely determines the regular part of the solution, w. More specifically, we obtain an extraction formula of λ by using the dual singular function and an extra cut-off function and then substitute it into (1.5) to obtain a single equation for w.

To this end, let

$$B(r_1) = \{(r, \theta) \colon 0 < r < r_1 \text{ and } 0 < \theta < \omega\} \cap \Omega$$

and

$$B(r_1; r_2) = \{(r, \theta) \colon r_1 < r < r_2 \text{ and } 0 < \theta < \omega\} \cap \Omega.$$

We consider a family of cut-off functions of r, $\eta_\rho(r)$, defined as follows:

(2.1)
$$\eta_\rho(r) = \begin{cases} 1 & \text{in } B(\frac{\rho R}{2}), \\ \frac{15}{16}\left\{\frac{8}{15} - \left(\frac{4r}{\rho R} - 3\right) + \frac{2}{3}\left(\frac{4r}{\rho R} - 3\right)^3 - \frac{1}{5}\left(\frac{4r}{\rho R} - 3\right)^5\right\} & \text{in } \bar{B}(\frac{\rho R}{2}; \rho R), \\ 0 & \text{in } \Omega \setminus \bar{B}(\rho R), \end{cases}$$

where ρ is a parameter in (0, 2], and $R \in \mathcal{R}$ is a fixed number so that the function $\eta_2 s$ vanishes identically on $\partial\Omega$. It is easy to check that $\eta_\rho \in C^2(\Omega)$ satisfies the following inequalities:

$$|\eta_\rho| \le 1, \quad |\partial_r \eta_\rho| \le \frac{C_1}{\rho R}, \quad \text{and} \quad |\partial_{rr} \eta_\rho| \le \frac{C_2}{(\rho R)^2} \tag{2.2}$$

with

$$C_1 = \frac{15}{4} \quad \text{and} \quad C_2 = \frac{40}{\sqrt{3}}, \tag{2.3}$$

where ∂_r and ∂_{rr} denote the respective first and second order partial differential operators with respect to r. Similarly, ∂_θ and $\partial_{\theta\theta}$ are the partial differential operators with respect to θ. Let

$$\eta^*(r) = \eta_2(r).$$

We use the standard notation and definition for the Sobolev spaces $H^t(B)$ for $t \geq 0$; the standard associated inner products are denoted by $(\cdot,\cdot)_{t,B}$, and their respective norms and seminorms are denoted by $\|\cdot\|_{t,B}$ and $|\cdot|_{t,B}$. We omit the subscript B from the inner product and norm designation when $B = \Omega$. For $t = 0$, $H^t(B)$ coincides with $L^2(B)$. In this case, the inner product and norm will be denoted by $(\cdot,\cdot)_B$ and $\|\cdot\|_B$ or $(\cdot,\cdot)$ and $\|\cdot\|$ when $B = \Omega$, respectively.

In order to get an extraction formula of λ, we will use the following lemma (see the proof of Lemma 8.4.3.1 in [16]), whose proof is provided in section 5 for completeness.

LEMMA 2.1. *For $\rho \in (0, 2]$, we have that*

$$(\Delta(\eta_\rho s), s_-) = -\pi. \tag{2.4}$$

Here and thereafter, we choose that $\eta = \eta_\rho$ in (1.4) and assume that $0 < \rho \leq 1$. That is, the singular function representation of the solution of problem (1.1) has the form

$$u = w + \lambda \eta_\rho s, \tag{2.5}$$

where $w \in H_0^1(\Omega) \cap H^2(\Omega)$ satisfies

$$-\Delta w - \lambda \Delta(\eta_\rho s) = f \quad \text{in } \Omega. \tag{2.6}$$

LEMMA 2.2. *The stress intensity factor λ can be expressed in terms of w by the following extraction formula:*

$$\lambda = \frac{1}{\pi}(w, \Delta(\eta^* s_-))_{B(R;2R)} + \frac{1}{\pi}(f, \eta^* s_-)_{B(2R)}. \tag{2.7}$$

Proof. Multiplying both sides of (2.6) by $\eta^* s_-$ and integrating over the domain Ω give that

$$\begin{aligned}(f, \eta^* s_-) &= -(\Delta w, \eta^* s_-) - \lambda(\Delta(\eta_\rho s), \eta^* s_-) \\ &= -(\Delta w, \eta^* s_-) - \lambda(\Delta(\eta_\rho s), s_-).\end{aligned}$$

The last equality used the fact that η^* equals one on the support of $\Delta(\eta_\rho s)$. Since $\eta^* s_- \in L^p(\Omega)$ for all $p < \frac{2\omega}{\pi}$, there exists a $p_0 > 2$ such that $\eta^* s_- \in L^{p_0}(\Omega)$ for a given $\pi < \omega < 2\pi$. Obviously, $\Delta(\eta^* s_-) \in L^{p_0}(\Omega)$ and $w \in W_{q_0}^2(\Omega)$, whose second order derivative belongs to $L^{q_0}(\Omega)$, such that $\frac{1}{p_0} + \frac{1}{q_0} = 1$. From Theorem 1.5.3.6 in [16] we have that

$$(\Delta w, \eta^* s_-) = (w, \Delta(\eta^* s_-))$$

since boundary terms vanish. Equality (2.7) now follows from Lemma 2.1 and the facts that supports of $\Delta(\eta^* s_-)$ and $\eta^* s_-$ are $B(R;2R)$ and $B(2R)$, respectively. □

Substituting the extraction formula, (2.7), of λ into (2.6) leads to

$$-\Delta w - \frac{1}{\pi}(w, \Delta(\eta^* s_-))_{B(R;2R)}\Delta(\eta_\rho s) = f + \frac{1}{\pi}(f, \eta^* s_-)_{B(2R)}\Delta(\eta_\rho s) \quad \text{in } \Omega. \tag{2.8}$$

Then its variational formulation is to find $w \in H_0^1(\Omega)$ such that

$$a(w,v) = g(v) \quad \forall\, v \in H_0^1(\Omega), \tag{2.9}$$

where the bilinear and linear forms are, respectively, given by

$$a(w,v) = (\nabla w, \nabla v) + \frac{1}{\pi}(w, \Delta(\eta^* s_-))_{B(R;2R)}(\nabla(\eta_\rho s), \nabla v)_{B(\rho R)} \tag{2.10}$$

and

$$g(v) = (f,v) - \frac{1}{\pi}(f, \eta^* s_-)_{B(2R)}(\nabla(\eta_\rho s), \nabla v)_{B(\rho R)}. \tag{2.11}$$

Note that the second terms in the respective bilinear and linear forms provide a singular correction so that $w \in H^2(\Omega)$. Note also that the bilinear form $a(\cdot,\cdot)$ is not symmetric. In the next section, we will establish the well-posedness of problem (2.9) for any $0 < \rho \le 1$.

3. Well-posedness. In this section, we establish the well-posedness of problem (2.9) by the use of the Fredholm alternative in $H_0^1(\Omega)$ for $0 < \rho \le 1$ in the definition of η_ρ. To do so, it is essential to estimate upper bounds of $\eta^* s_-$ and $\eta_\rho s$ on $B(R;2R)$ and $B(\rho R)$, respectively. These are listed in the following lemma whose proof is provided in section 5 since it involves lengthy computations.

LEMMA 3.1. *For any $0 < \rho \le 1$, we have that*

$$\|\Delta(\eta^* s_-)\|_{B(R;2R)} \le C_4 R^{-\frac{\pi}{\omega}-1} \tag{3.1}$$

with $C_4 = \sqrt{\frac{120\pi}{7}}$ and that

$$\|\eta_\rho s\|_{B(\rho R)} \le C_5 (\rho R)^{1+\frac{\pi}{\omega}} \quad \text{and} \quad \|\nabla(\eta_\rho s)\|_{B(\rho R)} \le C_6 (\rho R)^{\frac{\pi}{\omega}} \tag{3.2}$$

with $C_5 = \frac{\omega}{2\sqrt{\pi+\omega}}$ and $C_6 = \left(\frac{C_1^2\omega^2}{4(\pi+\omega)}(1 - 2^{-2(\frac{\pi}{\omega}+1)}) + \frac{\pi}{2}\right)^{\frac{1}{2}} < 3.81$.

We will need the following well-known Poincaré–Friedrichs inequality:

$$\|v\| \le C_\Omega \|\nabla v\| \quad \forall\, v \in H_0^1(\Omega), \tag{3.3}$$

where C_Ω is a positive constant depending only on the domain Ω. We especially have the following estimate of $C_{B(R;2R)}$.

LEMMA 3.2. *For any $v \in H_0^1(\Omega)$, we have that*

$$\|v\|_{B(R;2R)} \le \frac{2\omega R}{\pi}\|\nabla v\|. \tag{3.4}$$

Proof. We proceed to show the validity of (3.4) for $v \in C_0^\infty(\Omega)$. Then (3.4) would follow for $v \in H_0^1(\Omega)$ by continuity. Since $v(r,0) = v(r,\omega) = 0$ for $R < r < 2R$, the one-dimensional Poincaré–Friedrichs inequality gives that

$$\int_0^\omega v^2(r,\theta)\,d\theta \le \frac{\omega^2}{\pi^2}\int_0^\omega v_\theta^2(r,\theta)\,d\theta,$$

where v_θ denotes the first order partial derivative with respect to θ. Similarly, v_r below denotes the first order partial derivative with respect to r. This, together with the fact that $\nabla v \cdot \nabla v = v_r^2 + \frac{1}{r^2} v_\theta^2$, implies that

$$\|v\|_{B(R;2R)} = \left(\int_R^{2R} \int_0^\omega v^2(r,\theta)\, d\theta\, r dr \right)^{\frac{1}{2}} \le \frac{\omega}{\pi} \left(\int_R^{2R} \int_0^\omega v_\theta^2(r,\theta)\, d\theta\, r dr \right)^{\frac{1}{2}}$$
$$\le \frac{\omega}{\pi} \left(\int_R^{2R} \int_0^\omega r^2 \nabla v \cdot \nabla v\, d\theta\, r dr \right)^{\frac{1}{2}} \le \frac{2\omega R}{\pi} \|\nabla v\|.$$

This completes the proof of the lemma. □

LEMMA 3.3. *For* $0 < \rho \le 1$, *the bilinear form* $a(\cdot,\cdot)$, *defined in* (2.10), *is continuous and coercive in* $H_0^1(\Omega)$; *i.e., there exist positive constants* α, K, *and* M *such that*

$$\alpha \|\phi\|_1^2 \le a(\phi,\phi) + K \|\phi\|^2 \tag{3.5}$$

for all $\phi \in H_0^1(\Omega)$ *and that*

$$a(\phi,\psi) \le M \|\phi\|_1 \|\psi\|_1 \tag{3.6}$$

for all ϕ *and* ψ *in* $H_0^1(\Omega)$.

Proof. It follows from the Cauchy–Schwarz inequality and Lemma 3.1 that for any ϕ and ψ in $H_0^1(\Omega)$

$$\begin{aligned} &\left| \frac{1}{\pi} (\phi, \Delta(\eta^* s_-))_{B(R;2R)} (\nabla(\eta_\rho s), \nabla\psi)_{B(\rho R)} \right| \\ &\le \frac{1}{\pi} \|\Delta(\eta^* s_-)\|_{B(R;2R)} \|\nabla(\eta_\rho s)\|_{B(\rho R)} \|\phi\|_{B(R;2R)} \|\nabla\psi\|_{B(\rho R)} \\ &\le \frac{C_4 C_6}{\pi R} \rho^{\frac{\pi}{\omega}} \|\phi\|_{B(R;2R)} \|\nabla\psi\|. \end{aligned} \tag{3.7}$$

As an immediate consequence of the Cauchy–Schwarz inequality, (3.7), and (3.4), inequality (3.6) is valid with $M = 1 + \frac{2C_4C_6\omega}{\pi^2} \rho^{\frac{\pi}{\omega}}$. By using (3.7) with $\psi = \phi$ and the ϵ-inequality, we have that, for any $\epsilon > 0$,

$$\begin{aligned} a(\phi,\phi) &\ge \|\nabla\phi\|^2 - \frac{C_4C_6}{\pi R} \rho^{\frac{\pi}{\omega}} \|\phi\| \, \|\nabla\phi\| \\ &\ge \left(1 - \frac{C_4C_6}{2\pi R} \rho^{\frac{\pi}{\omega}} \epsilon \right) \|\nabla\phi\|^2 - \frac{C_4C_6}{2\pi R\epsilon} \rho^{\frac{\pi}{\omega}} \|\phi\|^2. \end{aligned}$$

Choosing $\epsilon = \frac{\pi R}{C_4C_6} \rho^{-\frac{\pi}{\omega}}$ gives that

$$a(\phi,\phi) \ge \frac{1}{2} \|\nabla\phi\|^2 - \frac{1}{2} \left(\frac{C_4C_6}{\pi R} \rho^{\frac{\pi}{\omega}} \right)^2 \|\phi\|^2.$$

Now, (3.5) follows from the Poincaré–Friedrichs inequality in (3.3) with $\alpha = \frac{1}{2}(1 + C_\Omega^2)^{-1}$ and $K = \frac{1}{2} \left(\frac{C_4C_6}{\pi R} \rho^{\frac{\pi}{\omega}} \right)^2$. □

To establish the well-posedness of our variational problem (2.9), we will make use of the Fredholm alternative (see, e.g., [15]). To this end, consider the following bilinear form:

$$a_\mu(w, v) = a(w, v) + \mu(w, v)$$

for $\mu \ge 0$.

THEOREM 3.4. *For* $0 < \rho \le 1$, *we have that*
(1) *problem* (2.9) *has a unique solution* w *in* $H_0^1(\Omega) \cap H^2(\Omega)$;
(2) *there exists a positive constant* γ *such that*

$$\gamma \|\phi\|_1 \le \sup_{\psi \in H_0^1(\Omega)} \frac{a(\phi,\, \psi)}{\|\psi\|_1} \tag{3.8}$$

for any $\phi \in H_0^1(\Omega)$.

Proof. Let $T_\mu : H_0^1(\Omega) \longrightarrow H^{-1}(\Omega)$ be the corresponding operator of the bilinear form $a_\mu(\cdot,\, \cdot)$, where $H^{-1}(\Omega)$ is the dual space of $H_0^1(\Omega)$ with the standard dual norm denoted by $\|\cdot\|_{-1}$. It follows from Lemma 3.3 that T_μ is a regular operator (i.e., it is one-to-one and onto and its inverse is bounded) for $\mu \ge K$ and its Fredholm index is zero. Since the index is independent of μ, T_0 satisfies the Fredholm alternative. That is, either T_0 is regular or $T_0 w = 0$ has a nontrivial solution. We will show that the second case does not hold. Now, the existence and uniqueness of the solution of problem (2.9) are a direct consequence of the facts that T_0 is regular and that g is in $H^{-1}(\Omega)$. To prove that $T_0 w = 0$ has only zero solution in $H_0^1(\Omega)$, we note that

$$a(w, v) = (T_0 w, v) = 0 \quad \forall\ v \in H_0^1(\Omega),$$

which implies that $w + \frac{1}{\pi}(w,\, \Delta(\eta^* s_-))\eta_\rho s$ is identically zero on $\bar{\Omega}$ since it satisfies the Poisson equation with zero data. Hence, w is a multiple of $\eta_\rho s$, but

$$(\eta_\rho s,\, \Delta(\eta^* s_-)) = 0$$

by the construction of the cut-off functions η_ρ and η^*. Therefore, w is identically zero on $\bar{\Omega}$.

To show the validity of the inf-sup condition in (3.8), we use the fact that the inverse of T_0 is bounded; i.e., there exists a positive constant γ such that

$$\|T_0^{-1}\|_{H^{-1}(\Omega) \to H_0^1(\Omega)} \le \frac{1}{\gamma}.$$

This implies that, for any $\phi \in H_0^1(\Omega)$,

$$\gamma \|\phi\|_1 \le \|T_0 \phi\|_{-1} = \sup_{\psi \in H_0^1(\Omega)} \frac{a(\phi,\, \psi)}{\|\psi\|_1},$$

which completes the proof of (3.8) and, hence, the theorem. ∎

COROLLARY 3.5. *Let* w *and* λ *be the solution of* (2.9) *and the stress intensity factor defined in* (2.7), *respectively. For* $0 < \rho \le 1$,

$$u = w + \lambda \eta_\rho s \tag{3.9}$$

is the solution of (1.1).

4. Finite element approximation. This section presents standard finite element approximation for w based on the variational problem in (2.9) and establishes error estimates in the L^2 and H^1 norms (see Theorem 4.2). Approximations to the stress intensity factor and the solution of problem (1.1) can then be computed according to (2.7) and (3.9), respectively. Their error estimates are, respectively, established in Theorems 4.3 and 4.4.

To this end, let $\mathcal{T}_h$ be a partition of the domain Ω into triangular finite elements; i.e., $\Omega = \cup_{K\in\mathcal{T}_h} K$ with $h = \max\{\operatorname{diam} K\colon K \in \mathcal{T}_h\}$. Assume that the triangulation $\mathcal{T}_h$ is regular. Let V_h be continuous piecewise linear finite element space; i.e.,

$$V_h = \{\phi_h \in C^0(\Omega)\colon \phi_h|_K \in P_1(K)\ \forall\ K \in \mathcal{T}_h,\ \phi_h = 0 \text{ on } \partial\Omega\} \subset H^1_0(\Omega),$$

where $P_1(K)$ is the space of linear functions on K. It is well known that

$$\inf_{\phi_h\in V_h} (\|\phi - \phi_h\| + h|\phi - \phi_h|_1) \le C_A h^{1+t} \|\phi\|_{1+t,\Omega} \tag{4.1}$$

for any $\phi \in H^1_0(\Omega) \cap H^{1+t}(\Omega)$ and $0 \le t \le 1$. Then the finite element approximation to problem (2.9) in $H^1_0(\Omega) \cap H^2(\Omega)$ becomes the following: find $w_h \in V_h$ such that

$$a(w_h, v) = g(v) \quad \forall\ v \in V_h. \tag{4.2}$$

In order to establish the error bound in the L^2 norm, we consider the following adjoint problem of (2.9) with a simplified linear form: find $z \in H^1_0(\Omega)$ such that

$$a(v, z) = (w - w_h, v) \quad \forall\ v \in H^1_0(\Omega). \tag{4.3}$$

The next lemma establishes the well-posedness of problem (4.3) and provides the regularity estimate for z.

LEMMA 4.1. *For $0 < \rho \le 1$, problem* (4.3) *has a unique solution z in $H^1_0(\Omega)$. Moreover, there is a singular function representation*

$$z = w_z + \lambda_z \eta_\rho s, \tag{4.4}$$

where $w_z \in H^2(\Omega) \cap H^1_0(\Omega)$ and $\lambda_z \in \mathcal{R}$ satisfy the regularity estimate

$$\|w_z\|_2 + |\lambda_z| \le C'_R \|w - w_h\|. \tag{4.5}$$

Proof. Similar proof as that of Theorem 3.4 shows that the adjoint problem in (4.3) has a unique solution in $H^1_0(\Omega)$ and that there exists a positive constant γ' such that

$$\gamma' \|\psi\|_1 \le \sup_{\phi\in H^1_0(\Omega)} \frac{a(\phi,\, \psi)}{\|\phi\|_1} \quad \forall\ \psi \in H^1_0(\Omega).$$

Let z be the solution of (4.3); by the Cauchy–Schwarz inequality we then have that

$$\|z\|_1 \le \frac{1}{\gamma'} \sup_{\phi\in H^1_0(\Omega)} \frac{a(\phi,\, z)}{\|\phi\|_1} = \frac{1}{\gamma'} \sup_{\phi\in H^1_0(\Omega)} \frac{(w - w_h,\, \phi)}{\|\phi\|_1} \le \frac{1}{\gamma'} \|w - w_h\|. \tag{4.6}$$

It is easy to check that the solution, $z \in H^1_0(\Omega)$, of problem (4.3) satisfies

$$\Delta z = \frac{1}{\pi} (\nabla(\eta_\rho s), \nabla z)_{B(\rho R)} \Delta(\eta^* s_-) - (w - w_h) \quad \text{in } \Omega. \tag{4.7}$$

Since the right-hand side of the above equation is at least in $L^2(\Omega)$, so is Δz. Therefore, z has the singular function representation

$$z = w_z + \lambda_z \eta_\rho s,$$

where $w_z \in H^2(\Omega) \cap H_0^1(\Omega)$ and

$$\|w_z\|_2 + |\lambda_z| \le C_R \|\Delta z\|.$$

Now, the regularity bound in (4.5) follows from the triangle and Cauchy–Schwarz inequalities, (4.6), and Lemma 3.1 that

$$\begin{aligned}\|w_z\|_2 + |\lambda_z| &\le C_R \|\Delta z\| \\ &\le C_R \left(\frac{1}{\pi} |(\nabla(\eta_\rho s), \nabla z)_{B(\rho R)}| \, \|\Delta(\eta^* s_-)\|_{B(R;2R)} + \|w - w_h\| \right) \\ &\le C_R \left(\frac{C_4 C_6}{\gamma' R \pi} \rho^{\frac{\pi}{\omega}} + 1 \right) \|w - w_h\|.\end{aligned}$$

This proves the inequality in (4.5) with

$$C_R' = C_R \left(\frac{C_4 C_6}{\gamma' R \pi} \rho^{\frac{\pi}{\omega}} + 1 \right)$$

and, hence, the lemma. □

Now we are ready to establish error bounds for the finite element approximation w_h in the L^2 and H^1 norms.

THEOREM 4.2. *For* $0 < \rho \le 1$*, there exists a positive constant* h_0 *such that for all* $h \le h_0$ (4.2) *has a unique solution* w_h *in* V_h*. Moreover, let* $w \in H^2(\Omega)$ *be the solution of* (2.9); *then we have the following error estimates:*

$$\|w - w_h\|_1 \le C_{11} h \|f\| \tag{4.8}$$

and

$$\|w - w_h\| \le C_{12}\, h^{1+\frac{\pi}{\omega}} \|f\|. \tag{4.9}$$

Proof. We first establish error bounds in (4.8) and (4.9) for any solution to problem (4.2) that may exist. Then, for $f \equiv 0$, the uniqueness of the solution to problem (2.9) and the error bound in (4.8) imply that $w_h \equiv 0$. Hence, (4.2) has a unique solution w_h in V_h since it is a finite dimensional problem with the same number of unknowns and equations.

To establish error bounds, note first the orthogonality property

$$a(w - w_h, v) = 0 \quad \forall\, v \in V_h. \tag{4.10}$$

By choosing $v = w - w_h$ in (4.3) and using the orthogonality property in (4.10) and the continuity bound in (3.6), we have that

$$\|w - w_h\|^2 = a(w - w_h, z) = a(w - w_h, z - I_h z) \le M\, \|w - w_h\|_1 \|z - I_h z\|_1, \tag{4.11}$$

where $I_h z \in V_h$ is the nodal interpolant of z. From the triangle inequality, approximation property (4.1), the fact that (see [4])

$$\|\eta_\rho s - I_h(\eta_\rho s)\|_1 \le C\, h^{\frac{\pi}{\omega}},$$

and Lemma 4.1, one has

$$\begin{aligned}\|z - I_h z\|_1 &\le \|w_z - I_h w_z\|_1 + |\lambda_z|\, \|\eta_\rho s - I_h(\eta_\rho s)\|_1 \\ &\le C\, h \|w_z\|_2 + C\, h^{\frac{\pi}{\omega}} |\lambda_z| \le C_D h^{\frac{\pi}{\omega}} \|w - w_h\|.\end{aligned}$$

Substituting this into (4.11) and dividing $\|w - w_h\|$ on both sides give

$$\|w - w_h\| \le MC_D h^{\frac{\pi}{\omega}} \|w - w_h\|_1. \tag{4.12}$$

Now, it follows from Lemma 3.3, orthogonality property (4.10), and inequality (4.12) that for any $v \in V_h$

$$\begin{aligned} \alpha\|w - w_h\|_1^2 &\le a(w - w_h,\, w - w_h) + K\|w - w_h\|^2 \\ &= a(w - w_h,\, w - v) + K\|w - w_h\|^2 \\ &\le M\|w - w_h\|_1\|w - v\|_1 + K\left(MC_D h^{\frac{\pi}{\omega}}\right)^2 \|w - w_h\|_1^2, \end{aligned}$$

which, together with approximation property (4.1), implies the validity of error bound (4.8) with $C_{11} = 2\alpha^{-1}MC_AC_R$ for all $h \le h_0$. Here,

$$h_0 = \left(\frac{\alpha}{2K(MC_D)^2}\right)^{\frac{\omega}{2\pi}}.$$

Error bound (4.9) is a direct consequence of (4.12) and (4.8) with $C_{12} = C_{11}MC_D$. We finish the proof of the theorem. □

The L^2 norm error bound for w obtained in (4.9) is not optimal and so are the error bounds for λ in the absolute value and for u in the L^2 norm (see (4.15) and (4.17)). Note that the L^2 error estimate in (4.9) is based on a simplified adjoint problem in (4.3), which does not have full regularity. Note also that the H^2 regularity estimate for w in (1.6) holds. At this stage, we are not sure if lack of approximation accuracy in the L^2 norm for w is because of that particular adjoint problem we used in our proof.

Now, approximations to the stress intensity factor and the solution of (1.1) can be computed according to (2.7) and (3.9) as follows:

$$\lambda_h = \frac{1}{\pi}(w_h, \Delta(\eta^* s_-))_{B(R;2R)} + \frac{1}{\pi}(f, \eta^* s_-)_{B(2R)} \tag{4.13}$$

and

$$u_h = w_h + \lambda_h \eta_\rho s, \tag{4.14}$$

respectively.

THEOREM 4.3. *Let λ be the stress intensity factor and λ_h its approximation defined in* (4.13). *Then*

$$|\lambda - \lambda_h| \le \frac{C_4}{\pi} R^{-\frac{\pi}{\omega}-1}\|w - w_h\| \le C_{13} R^{-\frac{\pi}{\omega}-1} h^{1+\frac{\pi}{\omega}} \|f\|. \tag{4.15}$$

Proof. Note from (2.7) and (4.13) that

$$\lambda - \lambda_h = \frac{1}{\pi}(w - w_h, \Delta(\eta^* s_-))_{B(R;2R)}.$$

Hence, (4.15) follows from the Cauchy–Schwarz inequality, Theorem 4.2, and Lemma 3.1 that

$$|\lambda - \lambda_h| \le \frac{1}{\pi}\|w - w_h\|\,\|\Delta(\eta^* s_-)\|_{B(R;2R)} \le C_{13} R^{-\frac{\pi}{\omega}-1} h^{1+\frac{\pi}{\omega}} \|f\|$$

with $C_{13} = \frac{C_{12}C_4}{\pi}$. □

THEOREM 4.4. *Let u be the solution of* (1.1) *and u_h its approximation defined in* (4.14); *then we have the following error estimates:*

$$\|u - u_h\|_1 \le C_{14} h \|f\| \tag{4.16}$$

and

$$\|u - u_h\| \le \left(1 + \frac{C_4 C_5}{\pi}\rho^{1+\frac{\pi}{\omega}}\right)\|w - w_h\| \le C_{15} h^{1+\frac{\pi}{\omega}}\|f\|. \tag{4.17}$$

Proof. It follows from (1.4) and (4.14) that

$$u - u_h = (w - w_h) + (\lambda - \lambda_h)\eta_\rho s.$$

By using the triangle inequality, Lemma 3.1, and Theorems 4.2 and 4.3, we have that

$$\begin{aligned}\|u - u_h\|_1 &\le \|w - w_h\|_1 + |\lambda - \lambda_h|\,\|\eta_\rho s\|_{1,B(\rho R)}\\ &\le C_{11} h\|f\| + C_{13}\rho^{\frac{\pi}{\omega}}(C_5\rho + C_6 R^{-1})h^{1+\frac{\pi}{\omega}}\|f\|.\end{aligned}$$

Therefore, (4.16) is valid with $C_{14} = C_{11} + C_{13}\rho^{\frac{\pi}{\omega}}(C_5\rho + C_6R^{-1})h^{\frac{\pi}{\omega}}$. In a similar fashion, by Lemma 3.1 and Theorems 4.2 and 4.3, we may prove the validity of (4.17) with $C_{15} = C_{12} + C_{13}C_5\rho^{1+\frac{\pi}{\omega}}$. This completes the proof of the theorem. ∎

5. Proofs of lemmas. In this section, we provide proofs for Lemmas 2.1 and 3.1. We will use both Cartesian and polar coordinates for convenience of calculations. The Laplacian and gradient operators especially take the following respective forms:

$$\Delta = \partial_{rr} + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_{\theta\theta} \quad \text{and} \quad \nabla = \begin{pmatrix}\cos\theta & -r^{-1}\sin\theta\\ \sin\theta & r^{-1}\cos\theta\end{pmatrix}\begin{pmatrix}\partial_r\\ \partial_\theta\end{pmatrix} \tag{5.1}$$

in polar coordinates.

Proof of Lemma 2.1. The proof of the lemma is similar to that of Lemma 8.4.3.1 in [16], but we provide the proof here for completeness. Let $\Omega_\varepsilon = \Omega\backslash\bar{B}(\varepsilon)$ for $0 < \varepsilon < \frac{\rho R}{2}$ and denote ν the outward unit normal on the boundary $\partial\Omega_\varepsilon$. Since s_- is a harmonic function, i.e., $\Delta s_- = 0$, and $\eta_\rho s = 0$ on $\partial\Omega$, it follows from integration by parts twice that

$$\begin{aligned}(\Delta(\eta_\rho s), s_-) &= \int_\Omega \Delta(\eta_\rho s)s_- dx = \lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon}\Delta(\eta_\rho s)s_- dx\\ &= \lim_{\varepsilon\to 0}\int_{\partial\Omega_\varepsilon}(\nu\cdot\nabla(\eta_\rho s))\,s_- ds - \lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon}\nabla(\eta_\rho s)\cdot\nabla s_- dx\\ &= \lim_{\varepsilon\to 0}\int_{\partial\Omega_\varepsilon}(\nu\cdot\nabla(\eta_\rho s))\,s_- ds - \lim_{\varepsilon\to 0}\int_{\partial\Omega_\varepsilon}\eta_\rho s\,(\nabla s_-\cdot\nu)\,ds\\ &= \lim_{\varepsilon\to 0}\int_{\partial\Omega_\varepsilon\cap\partial B_\varepsilon}(\nu\cdot\nabla(\eta_\rho s))\,s_- ds - \lim_{\varepsilon\to 0}\int_{\partial\Omega_\varepsilon\cap\partial B_\varepsilon}\eta_\rho s\,(\nabla s_-\cdot\nu)\,ds.\end{aligned}$$

The last equality follows from the fact that integrands are zero except on $\partial\Omega_\varepsilon\cap\partial B_\varepsilon = \{(r,\theta)\,:\, r = \varepsilon,\, 0 < \theta < \omega\}$. It is easy to see that

$$\nabla s = \begin{pmatrix}\cos\theta & -r^{-1}\sin\theta\\ \sin\theta & r^{-1}\cos\theta\end{pmatrix}\begin{pmatrix}\partial_r s\\ \partial_\theta s\end{pmatrix} = \frac{\pi}{\omega}r^{\frac{\pi}{\omega}-1}\begin{pmatrix}\sin(\frac{\pi}{\omega}-1)\theta\\ \cos(\frac{\pi}{\omega}-1)\theta\end{pmatrix}.$$

We then have that

$$(\nu \cdot \nabla(\eta_\rho s))s_- = (\nu \cdot \nabla s)\, s_- = -\left[\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix} \cdot \nabla s\right] s_- = -\frac{\pi}{\omega} r^{-1} \sin^2 \frac{\pi\theta}{\omega}$$

and that

$$\eta_\rho s(\nabla s_- \cdot \nu) = s(\nabla s_- \cdot \nu) = \frac{\pi}{\omega} r^{-1} \sin^2 \frac{\pi\theta}{\omega}.$$

Hence,

$$(\Delta(\eta_\rho s), s_-) = \lim_{\varepsilon\to 0} \left(-\frac{2\pi}{\omega}\right) \varepsilon \int_0^\omega \varepsilon^{-1} \sin^2 \frac{\pi\theta}{\omega} d\theta = -\pi.$$

This completes the proof of the lemma. □

Proof of Lemma 3.1. By the definition of η^*, we have that

$$\eta^*(r) = \begin{cases} 1 & \text{in } B(R),\\ \frac{15}{16}\left\{\frac{8}{15} - (\frac{2r}{R} - 3) + \frac{2}{3}\left(\frac{2r}{R} - 3\right)^3 - \frac{1}{5}\left(\frac{2r}{R} - 3\right)^5\right\} & \text{in } \bar{B}(R; 2R),\\ 0 & \text{in } \Omega \setminus \bar{B}(2R). \end{cases}$$

To show the validity of the inequality in (3.1), note first that s_- is harmonic and, hence, $\eta^* s_-$ is harmonic in $B(R)$. Thus,

$$\Delta(\eta^* s_-) = s_-\Delta\eta^* + 2\nabla\eta^* \cdot \nabla s_- = s_-\Delta\eta^* - \frac{2\pi}{\omega} r^{-(\frac{\pi}{\omega}+1)} \eta_r^* \sin\frac{\pi\theta}{\omega}. \tag{5.2}$$

Using polar coordinates, (5.2), and facts that

$$\int_0^\omega \sin^2\frac{\pi\theta}{\omega}\, d\theta = \frac{\omega}{2}, \quad \eta_r^* < 0 \text{ for } R < r < 2R, \quad \text{and} \quad \eta_{rr}^* \begin{cases} < 0, & R < r < \frac{3R}{2},\\ > 0, & \frac{3R}{2} < r < 2R, \end{cases}$$

we then have that

$$\begin{aligned}
&\|\Delta(\eta^* s_-)\|^2_{B(R;2R)} = \int_R^{2R} \int_0^\omega \left(s_-\Delta\eta^* - \frac{2\pi}{\omega} r^{-(\frac{\pi}{\omega}+1)} \eta_r^* \sin\frac{\pi\theta}{\omega}\right)^2 d\theta r dr\\
&= \frac{\omega}{2}\int_R^{2R} \left(\eta_{rr}^* + \frac{1}{r}\eta_r^*\right)^2 r^{-\frac{2\pi}{\omega}+1} dr - 2\pi \int_R^{2R} \left(\eta_{rr}^* + \frac{1}{r}\eta_r^*\right) \eta_r^* r^{-\frac{2\pi}{\omega}} dr\\
&\quad + \frac{2\pi^2}{\omega} \int_R^{2R} (\eta_r^*)^2 r^{-\frac{2\pi}{\omega}-1} dr\\
&= \frac{\omega}{2}\int_R^{2R} (\eta_{rr}^*)^2 r^{-\frac{2\pi}{\omega}+1} dr + (\omega - 2\pi) \int_R^{2R} \eta_{rr}^*\eta_r^* r^{-\frac{2\pi}{\omega}} dr\\
&\quad + \frac{\omega}{2}\left(1 - \frac{2\pi}{\omega}\right)^2 \int_R^{2R} (\eta_r^*)^2 r^{-\frac{2\pi}{\omega}-1} dr\\
&\le \frac{\omega}{2} R^{-\frac{2\pi}{\omega}+1} \int_R^{2R} (\eta_{rr}^*)^2 dr + (\omega - 2\pi)\left(\frac{3R}{2}\right)^{-\frac{2\pi}{\omega}} \int_R^{2R} \eta_{rr}^*\eta_r^* dr\\
&\quad + \frac{\omega}{2}\left(1 - \frac{2\pi}{\omega}\right)^2 R^{-\frac{2\pi}{\omega}-1} \int_R^{2R} (\eta_r^*)^2 dr\\
&\le \frac{5(4\pi^2 - 4\pi\omega + 13\omega^2)}{7\omega} R^{-\frac{2\pi}{\omega}-2} \le \frac{120\pi}{7} R^{-\frac{2\pi}{\omega}-2}.
\end{aligned}$$

Hence,

$$\|\Delta(\eta^* s_-)\|_{B(R;2R)} \le C_4 R^{-\frac{\pi}{\omega}-1},$$

where $C_4 = \sqrt{\frac{120\pi}{7}}$.

It follows from the fact that $|\eta_\rho| \le 1$ that

$$\|\eta_\rho s\|^2_{B(\rho R)} = \int_0^{\rho R}\int_0^{\omega} \eta_\rho^2 r^{1+\frac{2\pi}{\omega}} \sin^2 \frac{\pi\theta}{\omega}\, dr d\theta \le \frac{\omega^2}{4(\pi+\omega)}(\rho R)^{2(1+\frac{\pi}{\omega})},$$

which implies the first inequality in (3.2). We now proceed to show the validity of the second inequality in (3.2). By (5.1), it is easy to check that

$$\nabla(\eta_\rho s)\cdot\nabla(\eta_\rho s) = (\partial_r\eta_\rho)^2 s^2 + \left(\frac{\pi}{\omega}\right)^2 \eta_\rho^2 r^{2(\frac{\pi}{\omega}-1)} + \frac{2\pi}{\omega}\eta_\rho \partial_r \eta_\rho s r^{\frac{\pi}{\omega}-1}\sin\frac{\pi\theta}{\omega}.$$

Note the fact that

$$\nabla(\eta_\rho s)\cdot\nabla(\eta_\rho s) \le (\partial_r\eta_\rho)^2 s^2 + \left(\frac{\pi}{\omega}\right)^2 \eta_\rho^2 r^{2(\frac{\pi}{\omega}-1)}$$

since $\eta_\rho > 0$ and $\partial_r\eta_\rho < 0$ for all $(r,\theta)\in B(\rho R)$. It now follows from (2.2) and the fact that $\partial_r\eta_\rho = 0$ in $B(\frac{\rho R}{2})$ that

$$\begin{aligned}
\|\nabla(\eta_\rho s)\|^2_{B(\rho R)} &\le \int_0^{\rho R}\int_0^{\omega}\left((\partial_r\eta_\rho)^2 s^2 + \left(\frac{\pi}{\omega}\right)^2 \eta_\rho^2 r^{2(\frac{\pi}{\omega}-1)}\right) r\, d\theta dr \\
&= \frac{\omega}{2}\int_{\frac{\rho R}{2}}^{\rho R} (\partial_r\eta_\rho)^2 r^{\frac{2\pi}{\omega}+1}\, dr + \frac{\pi^2}{\omega}\int_0^{\rho R} \eta_\rho^2 r^{\frac{2\pi}{\omega}-1}\, dr \\
&\le \frac{\omega}{2}\left(\frac{C_1}{\rho R}\right)^2 \int_{\frac{\rho R}{2}}^{\rho R} r^{\frac{2\pi}{\omega}+1}\, dr + \frac{\pi^2}{\omega}\int_0^{\rho R} r^{\frac{2\pi}{\omega}-1}\, dr \\
&= \left(\frac{C_1^2\omega^2}{4(\pi+\omega)}\left(1-2^{-2(\frac{\pi}{\omega}+1)}\right) + \frac{\pi}{2}\right)(\rho R)^{\frac{2\pi}{\omega}}.
\end{aligned}$$

Thus,

$$\|\nabla(\eta_\rho s)\| \le C_6 (\rho R)^{\frac{\pi}{\omega}}$$

with

$$C_6 = \left(\frac{C_1^2\omega^2}{4(\pi+\omega)}\left(1-2^{-2(\frac{\pi}{\omega}+1)}\right) + \frac{\pi}{2}\right)^{\frac{1}{2}} < 3.81.$$

This completes the proof of the lemma. ∎

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