NUMERICAL LINEAR ALGEBRA WITH APPLICATIONS
Numer. Linear Algebra Appl. 2002; **9**:445–455 (DOI: 10.1002/nla.303)

A finite element method using singular functions for the Poisson equation: crack singularities

Zhiqiang Cai[1,*,†], Seokchan Kim[2,‡] and Gyungsoo Woo[2,§]

[1]*Department of Mathematics, Purdue University, 1395 Mathematical Science Building, West Lafayette, IN 47907-1395, U.S.A.*
[2]*Department of Applied Mathematics, Changwon National University, Changwon 641-773, Republic of Korea*

SUMMARY

In Cai and Kim (*SIAM J. Numer. Anal.* 2001; **39**:286), we developed and analysed a new accurate finite element method using singular functions for the Poisson equation on a two-dimensional polygonal domain with re-entrant corners. This method first computes the regular part of the solution, then stress intensity factors, and finally the solution itself. This note extends this method to the Poisson equation on a domain with cracks and considers a higher-order method when $f \in H^1(\Omega)$. Copyright © 2002 John Wiley & Sons, Ltd.

KEY WORDS: corner and crack singularities; finite element; stress intensity factor

1. INTRODUCTION

Solutions of elliptic boundary value problems defined on domains with corners have singular behaviour near the corners. This occurs even when data of the underlying problem are very smooth. Such singular behaviour affects the accuracy of the finite element method throughout the whole domain. For example, for the Poisson equation with homogeneous Dirichlet boundary conditions defined on a polygonal domain with re-entrant corners, it is well known that the solution has the singular function representation: $u = w + \sum_{j=1}^{J} \lambda_j \eta_j s_j$, where $w \in H^2(\Omega) \cap H_0^1(\Omega)$

*Correspondence to: Zhiqiang Cai, Department of Mathematics, Purdue University, 1395 Mathematical Science Building, West Lafayette, IN 47907-1395, U.S.A.
†E-mail: zcai@math.purdue.edu
‡E-mail: sckim@sarim.changwon.ac.kr
§E-mail: gswoo@sarim.changwon.ac.kr

Contract/grant sponsor: U.S. Department of Energy by Lawrence Livermore National Laboratory; contract/grant number: W-7405-Eng-48.
Contract/grant sponsor: KOSEF; contract/grant number: 1999-2-103-0002-3.
Contract/grant sponsor: Institute of Basic Science, Changwon National University

Received 14 October 2001
Revised 6 March 2002

is regular part of the solution, η_j are smooth cut-off functions, and s_j are known singular functions that depend only on the corresponding re-entrant angles. Coefficients λ_j can be expressed in terms of u by extraction formulas. Similar singular function representations hold for the solutions of interface, biharmonic, elasticity, and evolution problems (see References [1, 2]). In the context of mechanics, the coefficients λ_j are known as the stress intensity factors. Accurate calculation of these quantities is of great importance. For example, in aerodynamics, they characterize the lift of a flow around a corner shaped body (see Reference [3]). In fracture mechanics, most of the crack propagation criteria are expressed in terms of them (see Reference [4]).

Given a finite element approximation to the solution u, the stress intensity factors λ_j can be approximated using extraction formulas. It is well known (see Reference [5]) that u is in $H^r(\Omega)$ for $r<1+\pi/\omega$ where ω is the maximum of the re-entrant angles. Such lack of regularity affects the accuracy of the finite element approximation and, hence, the approximation to λ_j. In particular, standard continuous piecewise linear finite element on a quasi-uniform triangulation yields $O(h^{(\pi/\omega)-\varepsilon})$ and $O(h^{(2\pi/\omega)-\varepsilon})$ accuracy for any $\varepsilon>0$ in the H^1 and L^2 norms, respectively. In turn, this implies that the accuracy of the approximation to λ_j is $O(h^{(2\pi/\omega)-\varepsilon})$. There are several approaches in the literature for overcoming this difficulty (see References [6, 10–14] and references therein).

In Reference [7], we developed and analysed a new finite element method for the accurate computation of the solution and stress intensity factors. The loss of standard finite element approximation accuracy for elliptic boundary value problems with corner singularities is due to the non-smoothness of the solution. Therefore, it is natural to first approximate w, and then compute λ_j and u. To do so, we decouple the system by using the dual singular functions and extra cut-off functions with supports bigger than those of η_j. Now w is uniquely determined by a well-posed variational problem and λ_j can be expressed by an extraction formula in terms of w. Based on this variational problem, we showed that continuous piecewise linear finite element approximation on a quasi-uniform triangulation yields $O(h)$ optimal accuracy for w and u in H^1. Also, we established $O(h^{1+\pi/\omega})$ error bound for w and u in L^2 and for λ_j in the absolute value. Our numerical experiments in Reference [8] seem to indicate that our approach achieves $O(h^2)$ accuracy for u in L^2 and for λ_j in the absolute value.

The problem for w is no longer a nice Poisson equation. Instead, it is a Poisson equation perturbed by integral terms which are only non-zero on strips away from the corners. Because of such perturbation, the problem is non-symmetric and possibly indefinite. To solve non-symmetric algebraic equations arising from the discretization, it was shown in both theory and numerics in Reference [8] that a standard multigrid method is efficient. This is because the non-symmetric perturbation with pseudo-differential order of -1 is well-controlled by the Laplace operator whose pseudo-differential order is 2. The method adopted in Reference [8] is a V-cycle multigrid method that uses an exact coarsest grid solver and smoothing operators that depend only on the discrete Laplace operator.

The purpose of this note is to extend results in Reference [7] to a polygonal domain with cracks and consider a higher-order method when $f\in H^1(\Omega)$. The Poisson equation and the singular function representation are given in Section 2. A new extraction formula for λ_j and a variational problem for w are introduced in Section 3. Finite element methods and their error bounds are carried out in Section 4.

2. THE PROBLEM AND PRELIMINARIES

Consider the Poisson equations with homogeneous Dirichlet boundary conditions:

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned} \tag{1}$$

where f is a given function in either $L^2(\Omega)$ or $H^1(\Omega)$ and Ω is an open, bounded polygonal domain in $\mathscr{R}^2$. We use the standard notation and definition for the Sobolev spaces $H^t(B)$ for $t \geqslant 0$; the standard associated inner products are denoted by $(\cdot,\cdot)_{t,B}$, and their respective norms and seminorms are denoted by $\|\cdot\|_{t,B}$ and $|\cdot|_{t,B}$. The space $L^2(B)$ is interpreted as $H^0(B)$, in which case the inner product and norm will be denoted by $(\cdot,\cdot)_B$ and $\|\cdot\|_B$, respectively. We omit the subscript B from the inner product and norm designation when $B=\Omega$.

In the case that f is in $L^2(\Omega)$ but not in $H^1(\Omega)$, for simplicity, assume that the domain Ω has one re-entrant corner whose internal angle is $\omega = 2\pi$ (i.e. a crack). Extension to a domain with a finite number of cracks or re-entrant corners is straightforward. Let the corresponding vertex is at the origin. Define the *singular* and the *dual singular* functions by

$$s(r,\theta) = r^{\pi/\omega} \sin\frac{\pi\theta}{\omega} \quad \text{and} \quad s_-(r,\theta) = r^{-\pi/\omega}\sin\frac{\pi\theta}{\omega} \tag{2}$$

respectively, in the polar co-ordinates (r,θ). The co-ordinates are chosen at the origin so that the internal angle ω is spanned by the two half-lines $\theta=0$ and ω. Set

$$B_\omega(r_1;r_2) = \{(r,\theta)\colon\ r_1<r<r_2 \text{ and } 0<\theta<\omega\}\cap\Omega \quad \text{and} \quad B_\omega(r_1) = B(0;r_1)$$

Define a family of cut-off functions of r for a fixed ω, $\eta_\rho(r;\omega)$, as follows:

$$\eta_\rho(r;\omega) = \begin{cases} 1 & \text{in } B_\omega(\frac{\rho R}{2}) \\ \frac{15}{16}\{\frac{8}{15} - (\frac{4r}{\rho R}-3) + \frac{2}{3}(\frac{4r}{\rho R}-3)^3 - \frac{1}{5}(\frac{4r}{\rho R}-3)^5\} & \text{in } \bar{B}_\omega(\frac{\rho R}{2};\rho R) \\ 0 & \text{in } \Omega\backslash\bar{B}_\omega(\rho R) \end{cases} \tag{3}$$

where ρ is a parameter in $(0,2]$ and $R\in\mathscr{R}$ is a fixed number so that $\eta_2 s$ vanishes identically on $\partial\Omega$. It is well known (see, e.g. References [5, 9]) that the solution of problem (1) has the singular function representation

$$u = w^1 + \lambda\eta_\rho s \tag{4}$$

where $w^1\in H^{2-\varepsilon}(\Omega)\cap H_0^1(\Omega)$ for any $\varepsilon>0$ is regular part of the solution and $\lambda\in\mathscr{R}$ is the so-called *stress intensity factor*. Moreover, $w^1\in H^{2-\varepsilon}(\Omega)\cap H_0^1(\Omega)$ satisfies

$$-\Delta w^1 - \lambda\Delta(\eta_\rho s) = f \quad \text{in } \Omega \tag{5}$$

and the following regularity estimate:

$$\|w^1\|_{2-\varepsilon} \leqslant C_\varepsilon\|f\| \quad \text{and} \quad |\lambda| \leqslant C\|f\| \tag{6}$$

The stress intensity factor can be expressed in terms of u by the following *extraction formula* (see Reference [5]):

$$\lambda = \frac{1}{\pi}\left(\int_\Omega f\eta_\rho s_- \,\mathrm{d}x + \int_\Omega u\Delta(\eta_\rho s_-)\,\mathrm{d}x\right) \tag{7}$$

It is well known (cf Reference [5]) that the solution u of problem (1) is in $H^r(\Omega)$ for $r<1+\pi/\omega=\frac{3}{2}$. Such lack of regularity affects the accuracy of the finite element approximation and, hence, the approximation to the stress intensity factor.

In the case that $f\in H^1(\Omega)$, we also consider a second-order finite element method. Let $\omega_1,\dots,\omega_N$ be the internal angles of Ω satisfying $\pi/2<\omega_j\leqslant 2\pi$ and let v_j be the corresponding vertices. Let polar co-ordinates (r_j,θ_j) be chosen at the vertex v_j so that the internal angle ω_j is spanned by the two half-lines $\theta_j=0$ and ω_j. Let

$$\mathscr{L}_j = \left\{ l\in N:\ \frac{l\pi}{\omega_j}<2 \text{ and } \frac{l\pi}{\omega_j}\neq 1 \right\} \tag{8}$$

Note that $\mathscr{L}_j=\{1\}$ if $\pi/2<\omega_j<\pi$, $\mathscr{L}_j=\{1,2\}$ if $\pi<\omega_j\leqslant 3\pi/2$, $\mathscr{L}_j=\{1,2,3\}$ if $3\pi/2<\omega_j<2\pi$, and $\mathscr{L}_j=\{1,3\}$ if $\omega_j=2\pi$. Define the singular and dual singular functions by

$$s_{j,l}(r_j,\theta_j)=r_j^{l\pi/\omega_j}\sin\frac{l\pi}{\omega_j}\theta_j \quad\text{and}\quad s_{j,-l}(r_j,\theta_j)=r_j^{-l\pi/\omega_j}\sin\frac{l\pi}{\omega_j}\theta_j \tag{9}$$

for $l\in\mathscr{L}_j$, respectively. Then the solution of problem (1) admits the following singular function representation (see, e.g. References [5, 9]):

$$u=w^2+\sum_{j=1}^{N}\sum_{l\in\mathscr{L}_j}\lambda_{j,l}\eta_\rho(r_j;\omega_j)s_{j,l}(r_j,\theta_j) \tag{10}$$

where $w^2\in H^{3-\varepsilon}(\Omega)\cap H_0^1(\Omega)$ for any $\varepsilon>0$ satisfies

$$-\Delta w^2-\sum_{j=1}^{N}\sum_{l\in\mathscr{L}_j}\lambda_{j,l}\Delta(\eta_\rho s_{j,l})=f \quad \text{in } \Omega \tag{11}$$

Moreover, the following regularity estimate holds:

$$\|w^2\|_{3-\varepsilon}\leqslant C_\varepsilon\|f\|_1 \quad\text{and}\quad \sum_{j=1}^{N}\sum_{l\in\mathscr{L}_j}|\lambda_{j,l}|\leqslant C\|f\|_1 \tag{12}$$

The coefficients $\lambda_{j,l}$ can be expressed in terms of u by the following extraction formulas:

$$\lambda_{j,l}=\frac{1}{l\pi}\left(\int_\Omega f\eta_\rho s_{j,-l}\,\mathrm{d}x+\int_\Omega u\Delta(\eta_\rho s_{j,-l})\,\mathrm{d}x\right) \tag{13}$$

In subsequent sections, we assume that R in the definition of the cut-off functions is small enough so that the intersection of $B_{\omega_i}(\rho R)$ and $B_{\omega_j}(2R)$ for $i\neq j$ is empty.

Remark 2.1.
The stress intensity factors λ in (7) and $\lambda_{j,l}$ in (13) are independent of the choice of cut-off functions η_ρ.

3. WELL-POSED VARIATIONAL PROBLEM FOR w

This section derives well-posed variational problems for w^m $(m=1,2)$ (the regular part of the solution). The key step is to establish new extraction formulas for the stress intensity factors in terms of w^m so that Equations (5) and (11) involve only w^m (see (1) and (2) below). To do so, we use extra cut-off functions with a support bigger than that of η_ρ. Assume that $0<\rho\leqslant 1$ in both (4) and (10), then denote cut-off functions with bigger supports by

$$\eta^*(r;\omega)=\eta_2(r;\omega) \quad \text{or} \quad \eta^*(r_j;\omega_j)=\eta_2(r_j;\omega_j)$$

Lemma 3.1

(1) The stress intensity factor λ in (4) can be expressed in terms of w^1 by the following extraction formulas:

$$\lambda=\frac{1}{\pi}(w^1,\Delta(\eta^* s_-))_{B_\omega(R;2R)}+\frac{1}{\pi}(f,\eta^* s_-)_{B_\omega(2R)} \tag{14}$$

(2) The coefficients $\lambda_{j,l}$ in (13) can be expressed in terms of w^2 by the following extraction formulas:

$$\lambda_{j,l}=\frac{1}{l\pi}(w^2,\Delta(\eta^* s_{j,-l}))_{B_{\omega_j}(R;2R)}+\frac{1}{l\pi}(f,\eta^* s_{j,-l})_{B_{\omega_j}(2R)} \tag{15}$$

Proof
Since the proof for both (14) and (15) are same, we show only the validity of (15). Note that the proof presented here is much simpler than that in Reference [7]. Choosing $\eta_\rho(r_j;\omega_j)=\eta^*(r_j;\omega_j)$ in (13) (see Remark 2.1) gives

$$\lambda_{j,l}=\frac{1}{l\pi}(u,\Delta(\eta^* s_{j,-l}))+\frac{1}{l\pi}(f,\eta^* s_{j,-l})$$

Substituting (23), $u=w^2+\sum_{i=1}^N\sum_{k\in\mathscr{L}_i}\lambda_{i,k}\eta_\rho(r_i;\omega_i)s_{i,k}(r_i,\theta_i)$, into the above equation yields

$$\lambda_{j,l}=\frac{1}{l\pi}(w^2,\Delta(\eta^* s_{j,-l}))+\frac{1}{l\pi}(f,\eta^* s_{j,-l})+\frac{1}{l\pi}\sum_{i=1}^N\sum_{k\in\mathscr{L}_i}\lambda_{i,k}(\eta_\rho s_{i,k},\Delta(\eta^* s_{j,-l})) \tag{16}$$

It now suffices to show that the third term in (16) is equal to zero. When $i=j$, the support of $\eta_\rho(r_j;\omega_j)$ for $0<\rho\leqslant 1$ is $B_{\omega_j}(\rho R)$ on which $\eta^* s_{j,-l}$ is harmonic. Hence, for all $k,l\in\mathscr{L}_j$,

$$(\eta_\rho s_{j,k},\Delta(\eta^* s_{j,-l}))=0$$

When $i\neq j$, by the assumption that $B_{\omega_i}(\rho R)\cap B_{\omega_j}(2R)=\emptyset$ we have that

$$(\eta_\rho s_{i,k},\Delta(\eta^* s_{j,-l}))=0 \quad \forall k,l\in\mathscr{L}_j$$

These imply the third term in (16) equals to zero and, hence, (15). □

To derive a well-posed variational problem for w^m $(m=1,2)$, we first substitute extraction formula (14) into Equation (18) to obtain an integro-differential equation on w^1:

$$-\Delta w^1-\frac{1}{\pi}(w^1,\Delta(\eta^* s_-))_{B_\omega(R;2R)}\Delta(\eta_\rho s)=f+\frac{1}{\pi}(f,\eta^* s_-)_{B_\omega(2R)}\Delta(\eta_\rho s) \quad \text{in } \Omega$$

Similarly, we have

$$-\Delta w^2 - \sum_{j=1}^{N}\sum_{l\in\mathscr{L}_j}\frac{1}{l\pi}(w^2,\Delta(\eta^* s_{j,-l}))_{B_{\omega_j}(R;2R)}\Delta(\eta_\rho s_{j,l})$$

$$= f + \sum_{j=1}^{N}\sum_{l\in\mathscr{L}_j}\frac{1}{l\pi}(f,\eta^* s_{j,-l})_{B_{\omega_j}(2R)}\Delta(\eta_\rho s_{j,l}) \quad \text{in } \Omega$$

Multiplying the above equations by a test function $v \in H_0^1(\Omega)$, integrating over Ω, and applying Green's formula lead to the following variational problem: finding $w^m \in H_0^1(\Omega)$ $(m=1,2)$ such that

$$a_m(w^m, v) = g_m(v) \quad \forall v \in H_0^1(\Omega) \tag{17}$$

where the bilinear and linear forms are, respectively, given by

$$\begin{aligned} a_1(w^1,v) &= (\nabla w^1,\nabla v) + \frac{1}{\pi}(w^1,\Delta(\eta^* s_-))_{B_\omega(R;2R)}(\nabla(\eta_\rho s),\nabla v)_{B_\omega(\rho R)} \\ a_2(w^2,v) &= (\nabla w^2,\nabla v) + \sum_{j=1}^{N}\sum_{l\in\mathscr{L}_j}\frac{1}{l\pi}(w^2,\Delta(\eta^* s_{j,-l}))_{B_{\omega_j}(R;2R)}(\nabla(\eta_\rho s_{j,l}),\nabla v)_{B_{\omega_j}(\rho R)} \end{aligned} \tag{18}$$

and

$$g_1(v) = (f,v) - \frac{1}{\pi}(f,\eta^* s_-)_{B_\omega(2R)}(\nabla(\eta_\rho s),\nabla v)_{B_\omega(\rho R)} \tag{19}$$

$$g_2(v) = (f,v) - \sum_{j=1}^{N}\sum_{l\in\mathscr{L}_j}\frac{1}{l\pi}(f,\eta^* s_{j,-l})_{B_{\omega_j}(2R)}(\nabla(\eta_\rho s_{j,l}),\nabla v)_{B_{\omega_j}(\rho R)} \tag{20}$$

Note that the second terms in the respective bilinear and linear forms provide a singular correction so that $w^1 \in H^{2-\varepsilon}(\Omega)$ or $w^2 \in H^{3-\varepsilon}(\Omega)$ for $f \in H^1(\Omega)$. Note also that the bilinear forms $a_m(\cdot,\cdot)$ are not symmetric. In a similar fashion as Reference [7], we can prove the coercivity and continuity of the bilinear forms $a_m(\cdot,\cdot)$ and the well posedness of problem (17). We omit proofs of Lemma 3.2 and Theorem 3.1 below because they are similar to those of Lemma 3.3 and Theorem 3.4 in Reference [7], respectively.

Lemma 3.2
For $0<\rho\leqslant 1$ and $m=1,2$, the bilinear forms $a_m(\cdot,\cdot)$ are continuous and coercive in $H_0^1(\Omega)$; i.e. there exist positive constants α, K, and M such that

$$\alpha\|\phi\|_1^2 \leqslant a_m(\phi,\phi) + K\|\phi\|^2 \tag{21}$$

for all $\phi \in H_0^1(\Omega)$ and that

$$a_m(\phi,\psi) \leqslant M\|\phi\|_1\,\|\psi\|_1 \tag{22}$$

for all ϕ and ψ in $H_0^1(\Omega)$.

Theorem 3.1
For $0<\rho\leqslant 1$ and any $\varepsilon>0$, we have that

(1) if $f\in L^2(\Omega)$, then problem (4) with $m=1$ has a unique solution w^1 in $H_0^1(\Omega)\cap H^{2-\varepsilon}(\Omega)$;
(2) if $f\in H^1(\Omega)$, then problem (4) with $m=2$ has a unique solution w^2 in $H_0^1(\Omega)\cap H^{3-\varepsilon}(\Omega)$;
(3) there exists a positive constant γ such that

$$\gamma\|\phi\|_1\leqslant \sup_{\psi\in H_0^1(\Omega)}\frac{a_m(\phi,\psi)}{\|\psi\|_1} \tag{23}$$

for any $\phi\in H_0^1(\Omega)$.

4. FINITE ELEMENT APPROXIMATION

This section presents standard finite element approximation on a quasi-uniform grid for w^m based on the variational problem in (27). Approximations to the stress intensity factors and the solution of problem (24) can then be computed according to either (24) and (27) or (25) and (33) if $f\in H^1(\Omega)$, respectively. Error estimates are established in Theorems 4.1 and 4.2.

To this end, let $\mathscr{T}_h$ be a partition of the domain Ω into quasi-uniform triangular finite elements; i.e. $\Omega=\bigcup_{K\in\mathscr{T}_h}K$ with $h=\max\{\operatorname{diam}K:\ K\in\mathscr{T}_h\}$. Assume that the triangulation $\mathscr{T}_h$ is regular. Let $P_k(K)$ for $k=1,2$ be the set of all polynomials of the degree not greater than k. Denote continuous piecewise linear and quadratic finite element spaces by

$$V_h^k=\{\phi_h\in C^0(\Omega):\ \phi_h|_K\in P_k(K),\ \forall K\in\mathscr{T}_h,\ \phi_h=0 \text{ on } \partial\Omega\}\subset H_0^1(\Omega)$$

for $k=1,2$, respectively. It is well known that

$$\inf_{\phi_h\in V_h^k}(\|\phi-\phi_h\|+h|\phi-\phi_h|_1)\leqslant C_A h^{1+t}\|\phi\|_{1+t,\Omega} \tag{24}$$

for any $\phi\in H_0^1(\Omega)\cap H^{1+t}(\Omega)$ and $0\leqslant t<k$.

The finite element approximation to problem (27) with $m=1$ is to find $w_h^1\in V_h^1$ such that

$$a_1(w_h^1,v)=g_1(v)\quad \forall v\in V_h^1 \tag{25}$$

Approximations to the stress intensity factor in (24) and the solution u in (24) can then be computed by

$$\lambda_h=\frac{1}{\pi}(w_h^1,\Delta(\eta^* s_-))_{B_\omega(R;2R)}+\frac{1}{\pi}(f,\eta^* s_-)_{B_\omega(2R)} \tag{26}$$

and

$$u_h^1=w_h^1+\lambda_h\eta_\rho s \tag{27}$$

respectively. By the same proof as that in Reference [7], we have the following error bounds. Note that we lose $O(h^\varepsilon)$ order of accuracy in the crack case. This is because the regular part of the solution is no longer in $H^2(\Omega)$.

Theorem 4.1

(1) For $0<\rho\leqslant 1$, there exists a positive constant h_0 such that for all $h\leqslant h_0$ (25) has a unique solution w_h^1 in V_h^1. Moreover, let $w^1\in H^{2-\varepsilon}(\Omega)$ be the solution of (27) with $m=1$, then we have the following error estimates:

$$\|w^1-w_h^1\|_1\leqslant C_\varepsilon h^{1-\varepsilon}\|f\| \quad \text{and} \quad \|w^1-w_h^1\|\leqslant C_\varepsilon h^{3/2-\varepsilon}\|f\| \tag{28}$$

(2) Let λ and λ_h be defined in (24) and (26), respectively, then

$$|\lambda-\lambda_h|\leqslant C_\varepsilon h^{3/2-\varepsilon}\|f\| \tag{29}$$

(3) Let u be the solution of (24) and u_h^1 be its approximation defined in (27), then

$$\|u-u_h^1\|_1\leqslant C_\varepsilon h^{1-\varepsilon}\|f\| \quad \text{and} \quad \|u-u_h^1\|\leqslant C_\varepsilon h^{3/2-\varepsilon}\|f\| \tag{30}$$

In the case that $f\in H^1(\Omega)$, we have the regular part of the solution belonging to $H^{3-\varepsilon}(\Omega)$. Hence, we consider a quadratic finite element approximation: find $w_h^2\in V_h^2$ such that

$$a_2(w_h^2,v)=g_2(v) \quad \forall v\in V_h^2 \tag{31}$$

Approximations to the coefficients $\lambda_{j,l}$ and the solution u are approximated by

$$\lambda_{j,l}^h=\frac{1}{l\pi}(w_h^2,\Delta(\eta^* s_{j,-l}))_{B_{\omega_j}(R;2R)}+\frac{1}{l\pi}(f,\eta^* s_{j,-l})_{B_{\omega_j}(2R)} \tag{32}$$

and

$$u_h^2=w_h^2+\sum_{j=1}^{N}\sum_{l\in\mathscr{L}_j}\lambda_{j,l}^h\eta_\rho(r_j;\omega_j)s_{j,l}(r_j,\theta_j) \tag{33}$$

respectively. In order to establish the error bound in L^2, we consider the following adjoint problem of (27) with a simplified linear form: find $z\in H_0^1(\Omega)$ such that

$$a_2(v,z)=(w^2-w_h^2,v) \quad \forall v\in H_0^1(\Omega) \tag{34}$$

Next lemma establishes the well posedness of problem (34) and provides the regularity estimate for z.

Lemma 4.1

For $0<\rho\leqslant 1$, problem (34) has a unique solution z in $H_0^1(\Omega)$. Moreover, there is a singular function representation

$$z=w^z+\sum_{j=1}^{N}\sum_{l\in\mathscr{L}_j}\lambda_{j,l}^z\eta_\rho(r_j;\omega_j)s_{j,l}(r_j,\theta_j) \tag{35}$$

where $w^z\in H^{3-\varepsilon}(\Omega)\cap H_0^1(\Omega)$ for any $\varepsilon>0$ and $\lambda_{j,l}^z\in\mathscr{R}$ satisfy the regularity estimate:

$$\|w^z\|_{3-\varepsilon}\leqslant C_\varepsilon\|w^2-w_h^2\|_1 \quad \text{and} \quad \sum_{j=1}^{N}\sum_{l\in\mathscr{L}_j}|\lambda_{j,l}^z|\leqslant C\|w^2-w_h^2\|_1 \tag{36}$$

Proof
Similar to Theorem 3.1, the adjoint problem in (34) has a unique solution in $H_0^1(\Omega)$ and there exists a positive constant γ' such that

$$\gamma'\|\psi\|_1 \leqslant \sup_{\phi\in H_0^1(\Omega)} \frac{a_2(\phi,\psi)}{\|\phi\|_1} \quad \forall\psi\in H_0^1(\Omega)$$

Let z be the solution of (34), by the Cauchy–Schwarz inequality we then have that

$$\|z\|_1 \leqslant \frac{1}{\gamma'} \sup_{\phi\in H_0^1(\Omega)} \frac{a_2(\phi,z)}{\|\phi\|_1} = \frac{1}{\gamma'} \sup_{\phi\in H_0^1(\Omega)} \frac{(w^2-w_h^2,\phi)}{\|\phi\|_1} \leqslant \frac{1}{\gamma'}\|w^2-w_h^2\| \tag{37}$$

It is easy to check that the solution, $z\in H_0^1(\Omega)$, of problem (34) satisfies

$$\Delta z = \sum_{j=1}^{N}\sum_{l\in\mathscr{L}_j} \frac{1}{l\pi}(\nabla(\eta_\rho s_{j,l}),\nabla z)\Delta(\eta^* s_{j,-l}) - (w^2-w_h^2) \quad \text{in } \Omega \tag{38}$$

Since the right-hand side of the above equation is at least in $H^1(\Omega)$, so is Δz. Therefore, z has the singular function representation

$$z = w^z + \sum_{j=1}^{N}\sum_{l\in\mathscr{L}_j} \lambda_{j,l}^z \eta_\rho(r_j;\omega_j) s_{j,l}(r_j,\theta_j),$$

where $w^z\in H^{3-\varepsilon}(\Omega)\cap H_0^1(\Omega)$ and

$$\|w^z\|_{3-\varepsilon} \leqslant C_\varepsilon\|\Delta z\|_1 \quad \text{and} \quad \sum_{j=1}^{N}\sum_{l\in\mathscr{L}_j} |\lambda_{j,l}^z| \leqslant C\|\Delta z\|_1$$

The triangle and Cauchy–Schwarz inequalities and (37) give that

$$\|\Delta z\|_1 \leqslant \sum_{j=1}^{N}\sum_{l\in\mathscr{L}_j} \frac{1}{l\pi}\|\nabla(\eta_\rho s_{j,l})\|\|\nabla z\|\|\Delta(\eta^* s_{j,-l})\|_1 + \|w^2-w_h^2\|_1 \leqslant C\|w^2-w_h^2\|_1$$

Combining the above inequalities implies the regularity bound in (36). □

Now we are ready to establish error bounds for the finite element approximation.

Theorem 4.2
(1) For $0<\rho\leqslant 1$, there exists a positive constant h_0 such that for all $h\leqslant h_0$ (31) has a unique solution w_h^2 in V_h^2. Moreover, let $w^2\in H^{3-\varepsilon}(\Omega)$ be the solution of (27) with $m=2$, then we have the following error estimates:

$$\|w^2-w_h^2\|_1 \leqslant C_\varepsilon h^{2-\varepsilon}\|f\|_1 \quad \text{and} \quad \|w^2-w_h^2\| \leqslant C_\varepsilon h^{\frac{5}{2}-\varepsilon}\|f\|_1 \tag{39}$$

(2) Let $\lambda_{j,l}$ and $\lambda_{j,l}^h$ be defined in (25) and (32), respectively, then

$$|\lambda_{j,l}-\lambda_{j,l}^h| \leqslant C_\varepsilon h^{5/2-\varepsilon}\|f\|_1 \tag{40}$$

(3) Let u be the solution of (24) and u_h^2 be its approximation defined in (33), then

$$\|u-u_h^2\|_1 \leqslant C_\varepsilon h^{2-\varepsilon}\|f\|_1 \quad \text{and} \quad \|u-u_h^2\| \leqslant C_\varepsilon h^{5/2-\varepsilon}\|f\|_1 \tag{41}$$

Proof
We first establish error bounds in (39) for any solution to problem (31) that may exist. Then, for $f \equiv 0$, the uniqueness of the solution to problem (27) and the error bound in (39) imply that $w_h^2 \equiv 0$. Hence, (31) has a unique solution w_h^2 in V_h^2 since it is a finite dimensional problem with the same number of unknowns and equations.

To establish error bounds, note first the orthogonality property

$$a_2(w^2 - w_h^2, v) = 0 \quad \forall v \in V_h^2 \tag{42}$$

By choosing $v = w^2 - w_h^2$ in equation (34) and using the orthogonality property in (42) and the continuity bound in (32), we have that

$$\|w^2 - w_h^2\|^2 = a_2(w^2 - w_h^2, z) = a_2(w^2 - w_h^2, z - I_h z) \leqslant M\|w^2 - w_h^2\|_1 \|z - I_h z\|_1 \tag{43}$$

where $I_h z \in V_h^2$ is the nodal interpolant of z. From the triangle inequality, approximation property (24), the fact that $\|\eta_\rho s_{j,l} - I_h(\eta_\rho s_{j,l})\|_1 \leqslant C\,h^{1/2}$, and Lemma 4.1, one has

$$\begin{aligned}\|z - I_h z\|_1 &\leqslant \|w^z - I_h w^z\|_1 + \sum_{j=1}^{N} \sum_{l \in \mathscr{L}_j} |\lambda_{j,l}^z| \|\eta_\rho s_{j,l} - I_h(\eta_\rho s_{j,l})\|_1 \\ &\leqslant Ch\|w^z\|_2 + Ch^{1/2} \sum_{j=1}^{N} \sum_{l \in \mathscr{L}_j} |\lambda_{j,l}^z| \leqslant Ch^{1/2}\|w^2 - w_h^2\|\end{aligned}$$

Substituting this into (43) and dividing $\|w^2 - w_h^2\|$ on both sides give

$$\|w^2 - w_h^2\| \leqslant C\,h^{1/2}\|w^2 - w_h^2\|_1 \tag{44}$$

Now, it follows from Lemma 3.2, orthogonality property (42), and inequality (44) that for any $v \in V_h^2$

$$\begin{aligned}\alpha\|w^2 - w_h^2\|_1^2 &\leqslant a_2(w^2 - w_h^2, w^2 - w_h^2) + K\|w^2 - w_h^2\|^2 \\ &= a_2(w^2 - w_h^2, w^2 - v) + K\|w^2 - w_h^2\|^2 \\ &\leqslant M\|w^2 - w_h^2\|_1 \|w^2 - v\|_1 + CKh\|w^2 - w_h^2\|_1^2\end{aligned}$$

which implies that for sufficiently small $h < h_0$,

$$\|w^2 - w_h^2\|_1 \leqslant C\|w^2 - v\|_1 \quad \forall v \in V_h^2$$

Using approximation property (24) and regularity estimate (35) lead to the first error bound in (39) which, together with (44), implies the second error bound in (39).

It follows from (25) and (32) that

$$\lambda_{j,l} - \lambda_{j,l}^h = \frac{1}{l\pi}(w^2 - w_h^2, \Delta(\eta^* s_{j,-l}))_{B_{\omega_j}(R;2R)}$$

Equation (40) is then a direct consequence of the Cauchy–Schwarz inequality and (39). The difference of (33) and (33) gives that

$$u - u_h^2 = (w^2 - w_h^2) + \sum_{j=1}^{N} \sum_{l \in \mathscr{L}_j} (\lambda_{j,l} - \lambda_{j,l}^h)\eta_\rho s_{j,l}$$

Now, (41) follows from the triangle inequality, (39), and (40). This completes the proof of the theorem. □

The L^2 norm error bound for w^2 obtained in (39) is not optimal, and so are the error bounds for $\lambda_{j,l}$ in the absolute value and for u in the L^2 norm (see (40) and (41)). This is probably because a simplified adjoint problem in (34) is used in our error analysis.

ACKNOWLEDGEMENTS

The work of the first author was performed in part under the auspices of the U.S. Department of Energy by Lawrence Livermore National Laboratory under contract no. W-7405-Eng-48.

The second author was partially supported by KOSEF under the grants 1999-2-103-0002-3.

The third author was partially supported by the Institute for Basic Science, Changwon National University, 2001.

REFERENCES

1. Kellogg RB. Singularities in interface problems. In *Symposium on Numerical Solutions of Partial Differential Equations*, vol. II. Academic Press: New York, 1971; 351–400.
2. Grisvard P. *Singularities in Boundary Value Problems.* Masson: Paris, Springer: Berlin, 1992.
3. Djaoua M. Equations Intégrales pour un Probleme Singulier dans le Plan. *These de Troisieme Cycle*, Universite Pierre et Marie Curie, Paris, 1977.
4. Bui HD, *Mécanique de la Rupture Fragile*. Masson: Paris, 1978.
5. Grisvard P. *Elliptic Problems in Nonsmooth Domains*. Pitman: Boston, MA, 1985.
6. Grisvard P, Wendland W, Whiteman JR (eds). Singularities and Constructive Methods for Their Treatment, Lecture Notes in Mathematics, No. 1121. Springer: Berlin, 1985.
7. Cai Z, Kim S, A finite element method using singular functions for the Poisson equation: corner singularities. *SIAM Journal on Numerical Analysis* 2001; **39**:286–299.
8. Cai Z, Kim S, Shin B-C. Solution methods for the Poisson equation: corner singularities. *SIAM Journal on Scientific Computing* 2001; **23**:672–682.
9. Dauge M. Elliptic Boundary Value Problems on Corner Domains, Lecture Notes in Mathematics, vol. 1341. Springer: Berlin, Heidelberg, 1988.
10. Babuska I, Kellogg RB, Pitkaranta J. Direct and inverse error estimates for finite elements with mesh refinements, *Numerische Mathematik* 1979; **33**:447–471.
11. Blum H, Dobrowolski M. On finite element methods for elliptic equations on domains with corners. *Computing* 1982; **28**:53–63.
12. Brenner SC. Multigrid methods for the computation of singular solutions and stress intensity factor I: corner singularities. *Mathematics of Computation* 1999; **68**:559–583.
13. Brenner SC, Sung L-Y. Multigrid methods for the computation of singular solutions and stress intensity factors II: crack singularities. *BIT* 1997; **37**:623–643.
14. Fix GJ, Gulati S, Wakoff GI. On the use of singular functions with finite elements approximations. *Journal of Computational Physics* 1973; **13**:209–228.

