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THE DISCRETE FIRST-ORDER SYSTEM LEAST SQUARES: THE SECOND-ORDER ELLIPTIC BOUNDARY VALUE PROBLEM*

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Abstract. In [Z. Cai, T. Manteuffel, and S. F. McCormick, *SIAM J. Numer. Anal.*, 34 (1997), pp. 425–454], an L^2-norm version of first-order system least squares (FOSLS) was developed for scalar second-order elliptic partial differential equations. A limitation of this approach is the requirement of sufficient smoothness of the original problem, which is used for the equivalence of spaces between $(H^1)^d$ and $H(\text{div}\,) \cap H(\text{curl})$-type, where $d = 2$ or 3 is the dimension. By directly approximating $H(\text{div}\,) \cap H(\text{curl})$-type space based on the Helmholtz decomposition, this paper develops a discrete FOSLS approach in two dimensions. Under general assumptions, we establish error estimates in the L^2 and H^1 norms for the vector and scalar variables, respectively. Such error estimates are optimal with respect to the required regularity of the solution. A preconditioner for the algebraic system arising from this approach is also considered.

Key words. least-squares discretization, multigrid, preconditioner, second-order elliptic problems

AMS subject classifications. 65F10, 65F30

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1. Introduction.

Recently, there has been substantial interest in the use of least-squares principles for numerical approximations of elliptic partial differential equations and systems (see the recent review article [1] and references therein). In [5], Cai, Manteuffel, and McCormick developed an L^2-norm version of first-order system least squares (FOSLS) for scalar second-order elliptic partial differential equations in $d = 2$ or 3 dimensions. It was shown that the homogeneous FOSLS functional is equivalent to a $\mathcal{V} \times H^1(\Omega)$ norm with $\mathcal{V} = H(\text{div}\,;\Omega) \cap H(\text{curl}\,A;\Omega)$ under general assumptions, where A is the diffusion coefficient and Ω is the domain of the underlying problem. Moreover, such a norm was shown to be in fact an $H^1(\Omega)^{d+1}$ norm under the assumption that the original problem is H^2-regular. This product H^1 equivalence means that the minimization process amounts to solving a loosely coupled system of Poisson-like scalar equations. This in turn implies that standard finite element discretization and standard multigrid solution methods admit optimal H^1-like performance.

The limitation of this L^2-norm FOSLS is the requirement of sufficient smoothness of the underlying problem. Such smoothness guarantees the equivalence of norms between $\mathcal{V}$ and $H^1(\Omega)^d$ so that it can be approximated by standard continuous finite element space as in [5]. In general, when the domain Ω is not smooth or not convex or the coefficient A is not continuous, these two spaces are not equivalent. In fact, $\mathcal{V}$ is equal to $H^1(\Omega)^d$ plus a finite-dimensional space which consists of singular functions associated with corners of the boundary and interfaces. Therefore, standard continuous finite element spaces are not good approximations to $\mathcal{V}$ in general. In this paper,

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we will construct an appropriate approximation space for $\mathcal{V}$ based on the Helmholtz decomposition. Since our approximation space is discontinuous and is not contained in $\mathcal{V}$, we then modify the FOSLS functional to accommodate such discontinuity and nonconformity of finite element spaces. An alternative for overcoming such a limitation is the inverse-norm version of FOSLS (see [2]), but at the expense of rather awkward norm evaluation requirements.

The paper is organized as follows. The second-order elliptic boundary value problem and the L^2-norm version of the FOSLS approach are introduced in section 2, along with some notations. The discrete FOSLS approach is developed in section 3, and its error estimate is established in section 4. In section 5, we discuss preconditioners for the resulting system of linear equations.

2. First-order system least squares (FOSLS). Let Ω be a bounded, open, and simply connected domain in $\Re^2$ with Lipschitz boundary $\partial\Omega$. We consider the following scalar second-order elliptic boundary value problem:

$$\left\{ \begin{array}{rcll} -\nabla\cdot(A\nabla p) + \mathbf{b}\cdot\nabla p + cp & = & f & \text{in} \quad \Omega, \\ p & = & 0 & \text{on} \quad \Gamma_D, \\ \mathbf{n}\cdot(A\nabla p) & = & 0 & \text{on} \quad \Gamma_N, \end{array} \right. \tag{2.1}$$

where the symbols $\nabla\cdot$ and ∇ stand for the divergence and gradient operators, respectively; A is a 2×2 symmetric matrix of functions in $L^\infty(\Omega)$; $\mathbf{b}$ and c are the respective vector and scalar of functions in $L^\infty(\Omega)$; $f \in L^2(\Omega)$ is a given scalar function; $\partial\Omega = \Gamma_D \cup \Gamma_N$ is the partition of the boundary of Ω; and $\mathbf{n}$ is the outward unit vector normal to the boundary. For simplicity, assume that both Γ_D and Γ_N are nonempty, with the obvious generalization to quotient spaces when one of them is empty in the subsequent sections. We assume that A is uniformly symmetric positive definite and scaled appropriately; that is, there exist positive constants

$$0 < \lambda \le 1 \le \Lambda$$

such that

$$\lambda \boldsymbol{\xi}^T\boldsymbol{\xi} \le \boldsymbol{\xi}^T A \boldsymbol{\xi} \le \Lambda \boldsymbol{\xi}^T\boldsymbol{\xi} \tag{2.2}$$

for all $\boldsymbol{\xi} \in \Re^2$ and almost all $x \in \bar{\Omega}$.

We use standard notation and definitions for the Sobolev spaces $H^s(\Omega)^2$, associated inner products $(\cdot,\,\cdot)_s$, and respective norms $\|\cdot\|_s$, $s \ge 0$. (We suppress the designation Ω on the inner products and norms because dependence on region is clear by context.) $H^0(\Omega)^2$ coincides with $L^2(\Omega)^2$, in which case the norm and inner product will be denoted by $\|\cdot\|$ and $(\cdot,\,\cdot)$, respectively. Define subspaces of $H^1(\Omega)$:

$$H^1_D(\Omega) = \{q \in H^1(\Omega) :\, q = 0 \text{ on } \Gamma_D\} \text{ and } H^1_N(\Omega) = \{q \in H^1(\Omega) :\, q = 0 \text{ on } \Gamma_N\}.$$

Let $H^{-1}_D(\Omega)$ denote the dual of $H^1_D(\Omega)$ with the norm defined by

$$\|\phi\|_{H^{-1}_D(\Omega)} = \sup_{0\ne\psi\in H^1_D(\Omega)} \frac{(\phi,\,\psi)}{\|\psi\|_1}.$$

Denote the curl operator in $\Re^2$ by

$$\nabla\times = (-\partial_2,\, \partial_1)$$

and its formal adjoint by

$$\nabla^{\perp} = \begin{pmatrix} \partial_2 \\ -\partial_1 \end{pmatrix}.$$

Let

$$H(\mathrm{div}\, A^{\frac{1}{2}}; \Omega) = \{\mathbf{v} \in L^2(\Omega)^2 : \nabla \cdot (A^{\frac{1}{2}} \mathbf{v}) \in L^2(\Omega)\}$$

and

$$H(\mathrm{curl}\, A^{-\frac{1}{2}}; \Omega) = \{\mathbf{v} \in L^2(\Omega)^2 : \nabla\times (A^{-\frac{1}{2}} \mathbf{v}) \in L^2(\Omega)\},$$

which are Hilbert spaces under norms

$$\|\mathbf{v}\|_{H(\mathrm{div}\ A^{\frac{1}{2}};\Omega)} = \left(\|\mathbf{v}\|^2 + \left\| \nabla \cdot \left(A^{\frac{1}{2}} \mathbf{v} \right) \right\|^2 \right)^{\frac{1}{2}}$$

and

$$\|\mathbf{v}\|_{H(\mathrm{curl}\, A^{-\frac{1}{2}};\Omega)} = \left(\|\mathbf{v}\|^2 + \left\| \nabla\times \left(A^{-\frac{1}{2}} \mathbf{v} \right) \right\|^2 \right)^{\frac{1}{2}},$$

respectively. When A is the identity matrix, we use the simpler notations $H(\mathrm{div}; \Omega)$ and $H(\mathrm{curl}; \Omega)$. Define the subspaces

$$H_0(\mathrm{div}\, A^{\frac{1}{2}}; \Omega) = \{\mathbf{v} \in H(\mathrm{div}\, A^{\frac{1}{2}}; \Omega) : \mathbf{n} \cdot (A^{\frac{1}{2}} \mathbf{v}) = 0 \text{ on } \Gamma_N\},$$

$$H_0(\mathrm{curl}\, A^{-\frac{1}{2}}; \Omega) = \{\mathbf{v} \in H(\mathrm{curl}\, A^{-\frac{1}{2}}; \Omega) : \boldsymbol{\tau} \cdot (A^{-\frac{1}{2}} \mathbf{v}) = 0 \text{ on } \Gamma_D\},$$

and denote

$$\mathcal{U} = H_0(\mathrm{div}\, A^{\frac{1}{2}}; \Omega) \cap H_0(\mathrm{curl}\, A^{-\frac{1}{2}}; \Omega),$$

where $\boldsymbol{\tau}$ represents the unit vector tangent to the boundary oriented counterclockwise.

Introducing an independent vector variable

$$\mathbf{u} = A^{\frac{1}{2}}\, \nabla p,$$

by using the homogeneous Dirichlet boundary condition on Γ_D we have that

$$\nabla\times (A^{-\frac{1}{2}} \mathbf{u}) = 0 \quad \text{in } \Omega \quad \text{and} \quad \boldsymbol{\tau} \cdot (A^{-\frac{1}{2}} \mathbf{u}) = 0 \quad \text{on } \Gamma_D.$$

Then an equivalent extended system for problem (2.1) is

$$\text{(2.3)} \qquad \left\{ \begin{array}{rcll} \mathbf{u} - A^{\frac{1}{2}} \nabla p &=& \mathbf{0} & \text{in} \quad \Omega, \\ -\nabla \cdot (A^{\frac{1}{2}} \mathbf{u}) + \mathbf{b} \cdot (A^{-\frac{1}{2}} \mathbf{u}) + cp &=& f & \text{in} \quad \Omega, \\ \nabla \times (A^{-\frac{1}{2}} \mathbf{u}) &=& 0 & \text{in} \quad \Omega, \\ p &=& 0 & \text{on} \quad \Gamma_D, \\ \mathbf{n} \cdot (A^{\frac{1}{2}} \mathbf{u}) &=& 0 & \text{on} \quad \Gamma_N, \\ \boldsymbol{\tau} \cdot (A^{-\frac{1}{2}} \mathbf{u}) &=& 0 & \text{on} \quad \Gamma_D. \end{array} \right.$$

Define the FOSLS functional as follows (see [5] or [6] for Poisson's equations):

$$G(\mathbf{v}, q; f) = \|\mathbf{v} - A^{\frac{1}{2}} \nabla q\|^2 + \|f + \nabla \cdot (A^{\frac{1}{2}} \mathbf{v}) - \mathbf{b} \cdot (A^{-\frac{1}{2}} \mathbf{v}) - cq\|^2 + \|\nabla \times (A^{-\frac{1}{2}} \mathbf{v})\|^2$$

for $(\mathbf{v}, q) \in \mathcal{U} \times H^1_D(\Omega)$. Then the FOSLS variational problem for (2.1) is to minimize the quadratic functional $G(\mathbf{v}, q; f)$ over $\mathcal{U} \times H^1_D(\Omega)$: find $(\mathbf{u}, q) \in \mathcal{U} \times H^1_D(\Omega)$ such that

$$\text{(2.4)} \qquad G(\mathbf{u}, p; f) = \inf_{(\mathbf{v},q) \in \mathcal{U} \times H^1_D(\Omega)} G(\mathbf{v}, q; f).$$

3. Discrete FOSLS. The least-squares approach defined in the previous section was proposed and analyzed in [5]. In particular, it was shown in [5] that the homogeneous functional is elliptic in the $H^1(\Omega)^3$ norm under certain H^2 regularity assumptions. This implies optimal H^1-like performance for standard finite element discretization and standard multigrid solution methods. An unfortunate limitation of this FOSLS approach is that this product H^1 equivalence generally requires sufficient smoothness of the original problem. Such a requirement is needed for the equivalence between the spaces $H^1(\Omega)^2$ and $\mathcal{U} = H_0(\text{div}\, A^{\frac{1}{2}};\Omega) \cap H_0(\text{curl}\, A^{-\frac{1}{2}};\Omega)$ and the quasi-optimality of finite element approximations in the H^1 norm for each variable. To overcome such a difficulty, we use discontinuous approximation spaces for the vector variable and modify this FOSLS functional to accommodate such a discontinuity of finite element spaces. Extension of the approach proposed in this section to the least-squares functional studied in [4, 8] is straightforward.

Discontinuous approximation spaces that we will employ are motivated by the following Helmholtz decomposition, for any $\mathbf{u} \in \mathcal{U}$:

$$\mathbf{u} = A^{\frac{1}{2}}\nabla s + A^{-\frac{1}{2}}\nabla^\perp t, \tag{3.1}$$

where $s \in H^1_D(\Omega)$ is the unique solution of

$$\begin{cases} \nabla\cdot(A\nabla s) = \nabla\cdot(A^{\frac{1}{2}}\mathbf{u}) & \text{in } \Omega, \\ s = 0 & \text{on } \Gamma_D, \\ \mathbf{n}\cdot(A\nabla s) = 0 & \text{on } \Gamma_N \end{cases}$$

and $t \in H^1_N(\Omega)$ is the unique solution of

$$\begin{cases} \nabla\times(A^{-1}\nabla^\perp t) = \nabla\times(A^{-\frac{1}{2}}\mathbf{u}) & \text{in } \Omega, \\ \boldsymbol{\tau}\cdot(A^{-1}\nabla^\perp t) = 0 & \text{on } \Gamma_D, \\ t = 0 & \text{on } \Gamma_N. \end{cases}$$

It is then natural to approximate the scalar functions $s \in H^1_D(\Omega)$ and $t \in H^1_N(\Omega)$ by standard continuous piecewise polynomials.

Let $\mathcal{T}_h$ be a partition of the domain Ω into finite elements; i.e., $\Omega = \cup_{K\in\mathcal{T}_h} K$ with $h = \max\{h_K = \text{diam}(K) : K \in \mathcal{T}_h\}$. Assume that the triangulation $\mathcal{T}_h$ is regular (see [7]). Let $\mathcal{P}^h_{m-1}$ be a finite-dimensional space consisting of continuous piecewise polynomials of degree at most $m-1$ with respect to the triangulation $\mathcal{T}_h$. Denote standard finite element spaces by

$$\mathcal{S}^h_D = H^1_D(\Omega) \cap \mathcal{P}^h_{m-1} \quad \text{and} \quad \mathcal{S}^h_N = H^1_N(\Omega) \cap \mathcal{P}^h_{m-1}$$

and define the approximation space for the vector variable by

$$\mathcal{U}^h = \left(A^{\frac{1}{2}}\nabla\mathcal{S}^h_D\right) \oplus \left(A^{-\frac{1}{2}}\nabla^\perp\mathcal{S}^h_N\right).$$

It is an immediate consequence of the integration by parts and homogeneous boundary conditions that two subspaces $A^{\frac{1}{2}}\nabla\mathcal{S}^h_D$ and $A^{-\frac{1}{2}}\nabla^\perp\mathcal{S}^h_N$ are orthogonal with respect to the L^2 inner product. That is,

$$(A^{\frac{1}{2}}\nabla s,\, A^{-\frac{1}{2}}\nabla^\perp t) = 0 \tag{3.2}$$

for any $s \in \mathcal{S}^h_D$ and any $t \in \mathcal{S}^h_N$.

Note that $\mathcal{U}^h$ is not contained in $\mathcal{U}$ and, hence, the FOSLS functional $G(\cdot\,;\cdot)$ defined in the previous section is not well defined on $\mathcal{U}^h \times \mathcal{S}_D^h$. Therefore, we need to replace the divergence and curl operators in the $G(\cdot\,;\cdot)$ by the corresponding discrete operators. To this end, define the discrete divergence operator, $\nabla_h\cdot :\ L^2(\Omega)^2 \longrightarrow \mathcal{S}_D^h$, for given $\mathbf{v} \in L^2(\Omega)^2$ by $\phi = \nabla_h \cdot \mathbf{v} \in \mathcal{S}_D^h$ satisfying

$$(\phi,\, q) = -(\mathbf{v},\, \nabla q) \quad \forall\ q \in \mathcal{S}_D^h$$

and the discrete curl operator, $\nabla_h\times\ :\ L^2(\Omega)^2 \longrightarrow \mathcal{S}_N^h$, for given $\mathbf{v} \in L^2(\Omega)^2$ by $\psi = \nabla_h \times \mathbf{v} \in \mathcal{S}_N^h$ satisfying

$$(\psi,\, q) = (\mathbf{v}, \nabla^{\perp} q) \quad \forall\ q \in \mathcal{S}_N^h.$$

Finally, we denote Q_h the L^2-projection operator onto $\mathcal{S}_D^h$.

Now, we are ready to define the discrete FOSLS functional:

$$\begin{aligned} G_h(\mathbf{v}, q; f) = \|\mathbf{v} - A^{\frac{1}{2}}\nabla q\|^2 + \|f + \nabla_h \cdot (A^{\frac{1}{2}}\mathbf{v}) - Q_h\big(\mathbf{b}\cdot(A^{-\frac{1}{2}}\mathbf{v})\big) - cq\|^2 \\ + \|\nabla_h \times (A^{-\frac{1}{2}}\mathbf{v})\|^2 \end{aligned}$$

for $(\mathbf{v}, q) \in \mathcal{U}^h\times\mathcal{S}_D^h$. Our discrete FOSLS finite element approximation for (2.1) is then to minimize the quadratic functional $G_h(\mathbf{v}, q; f)$ over $\mathcal{U}^h\times\mathcal{S}_D^h$: find $(\mathbf{u}_h,\, p_h) \in \mathcal{U}^h\times\mathcal{S}_D^h$ such that

$$G_h(\mathbf{u}_h,\, p_h; f) = \inf_{(\mathbf{v},q)\in\mathcal{U}^h\times\mathcal{S}_D^h} G_h(\mathbf{v},\, q; f). \tag{3.3}$$

Denote the norm over $\mathcal{U}^h \times \mathcal{S}_D^h$ by

$$|||(\mathbf{v}, q)||| = \Big(\|q\|_1^2 + \|\mathbf{v}\|^2 + \|\nabla_h \cdot (A^{\frac{1}{2}}\mathbf{v})\|^2 + \|\nabla_h \times (A^{-\frac{1}{2}}\mathbf{v})\|^2\Big)^{\frac{1}{2}}.$$

THEOREM 3.1. *The homogeneous functional $G_h(\cdot; 0)$ is uniformly elliptic and continuous in $\mathcal{U}^h \times \mathcal{S}_D^h$; i.e., for any $(\mathbf{v}, q) \in \mathcal{U}^h \times \mathcal{S}_D^h$, there exists a positive constant C such that*

$$\frac{1}{C}|||(\mathbf{v}, q)|||^2 \le G_h(\mathbf{v}, q; 0) \le C|||(\mathbf{v}, q)|||^2. \tag{3.4}$$

Proof. The upper bound in (3.4) is an immediate consequence of the triangle inequality and the boundedness of coefficients A, $\mathbf{b}$, c and the L^2-projection operator Q_h. To show the validity of the lower bound in (3.4), we first establish the following inequality: there exists a positive constant C such that

$$\frac{1}{C}|||(\mathbf{v}, q)|||^2 \le \tilde{G}(\mathbf{v}, q) \quad \forall\ (\mathbf{v}, q) \in \mathcal{U}^h \times \mathcal{S}_D^h, \tag{3.5}$$

where

$$\begin{aligned} \tilde{G}(\mathbf{v}, q) &= \|\mathbf{v} - A^{\frac{1}{2}}\nabla q\|^2 + \|\nabla_h \cdot (A^{\frac{1}{2}}\mathbf{v}) - Q_h(\mathbf{b}\cdot\nabla q) - cq\|^2 + \|\nabla_h \times (A^{-\frac{1}{2}}\mathbf{v})\|^2 \\ &= \left\| \begin{pmatrix} I & -A^{\frac{1}{2}}\nabla \\ \nabla_h \cdot A^{\frac{1}{2}} & -Q_h\mathbf{b}\cdot\nabla - cI \end{pmatrix} \begin{pmatrix} \mathbf{v} \\ q \end{pmatrix} \right\|^2 + \|\nabla_h \times (A^{-\frac{1}{2}}\mathbf{v})\|^2. \end{aligned}$$

Then the lower bound in (3.4) follows from the fact that

$$G_h(\mathbf{v},q;0) = \left\| \begin{pmatrix} I & 0 \\ -Q_h\mathbf{b}\cdot A^{-\frac{1}{2}} & I \end{pmatrix} \begin{pmatrix} I & -A^{\frac{1}{2}}\nabla \\ \nabla_h\cdot A^{\frac{1}{2}} & -Q_h\mathbf{b}\cdot\nabla - cI \end{pmatrix} \begin{pmatrix} \mathbf{v} \\ q \end{pmatrix} \right\|^2$$

$$+ \|\nabla_h \times (A^{-\frac{1}{2}}\mathbf{v})\|^2$$

and that the largest and smallest singular values of the transformation matrix

$$\begin{pmatrix} I & 0 \\ -Q_h\mathbf{b}\cdot A^{-\frac{1}{2}} & I \end{pmatrix}$$

are bounded.

To prove the validity of (3.5), let $Xq = Q_h(\mathbf{b}\cdot\nabla q) + cq$ for convenience. Since $\|Q_h\| = 1$, (2.2) and the triangle and Poincaré–Friedrichs inequalities yield

$$\|Xq\| \le C\|A^{\frac{1}{2}}\nabla q\|.$$

It now follows from the definition of the discrete divergence operator and the Cauchy–Schwarz inequality that

$$\begin{aligned} \|A^{\frac{1}{2}}\nabla q\|^2 &= (A^{\frac{1}{2}}\nabla q - \mathbf{v}, A^{\frac{1}{2}}\nabla q) + (A^{\frac{1}{2}}\mathbf{v}, \nabla q) \\ &= (A^{\frac{1}{2}}\nabla q - \mathbf{v}, A^{\frac{1}{2}}\nabla q) - \big(\nabla_h\cdot(A^{\frac{1}{2}}\mathbf{v}), q\big) \\ &= (A^{\frac{1}{2}}\nabla q - \mathbf{v}, A^{\frac{1}{2}}\nabla q) - \big(\nabla_h\cdot(A^{\frac{1}{2}}\mathbf{v}) - Xq, q\big) - (Xq,q) \\ &\le \|A^{\frac{1}{2}}\nabla q - \mathbf{v}\|\,\|A^{\frac{1}{2}}\nabla q\| + \|\nabla_h\cdot(A^{\frac{1}{2}}\mathbf{v}) - Xq\|\,\|q\| + \|Xq\|\,\|q\|, \end{aligned}$$

which, together with the Poincaré–Friedrichs inequality, implies that

$$\|A^{\frac{1}{2}}\nabla q\| \le C\left(\|A^{\frac{1}{2}}\nabla q - \mathbf{v}\| + \|\nabla_h\cdot(A^{\frac{1}{2}}\mathbf{v}) - Q_h(\mathbf{b}\cdot\nabla q) - cq\| + \|q\|\right). \tag{3.6}$$

The triangle and Poincaré–Friedrichs inequalities and (3.6) give that

$$|||(\mathbf{v},q)|||^2 \le C\left(\tilde{G}(\mathbf{v},q) + \|q\|^2\right).$$

Now, (3.5) is a consequence of the standard compactness argument. This completes the proof of the theorem. □

4. Error estimates. This section establishes error estimates in the L^2 norm for the vector variable and the H^1 norm for the scalar variable (see Theorem 4.1). Such error estimates are optimal with respect to the required regularity of the solution.

Let $(\mathbf{u}_h, p_h) \in \mathcal{U}^h \times \mathcal{S}_D^h$ be the solution of the discrete problem in (3.3). The corresponding variational form of (3.3) is to find $(\mathbf{u}_h, p_h) \in \mathcal{U}^h \times \mathcal{S}_D^h$ such that

(4.1)

$$b_h(\mathbf{u}_h, p_h; \mathbf{v}, q) = \big(f, -\nabla_h\cdot(A^{\frac{1}{2}}\mathbf{v}) + Q_h(\mathbf{b}\cdot(A^{-\frac{1}{2}}\mathbf{v})) + cq\big) \quad \forall\,(\mathbf{v},q) \in \mathcal{U}^h \times \mathcal{S}_D^h,$$

where the bilinear form $b_h(\cdot,\cdot)$ is induced from the quadratic form $G_h(\cdot;0)$:

$$\begin{aligned} b_h(\mathbf{u}_h, p_h; \mathbf{v}, q) &= \big(\mathbf{u}_h - A^{\frac{1}{2}}\nabla p_h, \mathbf{v} - A^{\frac{1}{2}}\nabla q\big) + \big(\nabla_h\times(A^{-\frac{1}{2}}\mathbf{u}_h), \nabla_h\times(A^{-\frac{1}{2}}\mathbf{v})\big) \\ &+\big(\nabla_h\cdot(A^{\frac{1}{2}}\mathbf{u}_h) - Q_h(\mathbf{b}\cdot(A^{-\frac{1}{2}}\mathbf{u}_h)) - cp_h, \nabla_h\cdot(A^{\frac{1}{2}}\mathbf{v}) - Q_h(\mathbf{b}\cdot(A^{-\frac{1}{2}}\mathbf{v})) - cq\big). \end{aligned} \tag{4.2}$$

To deduce the error equation, we need the following lemma.

LEMMA 4.1. *Let* $(\mathbf{u}, p) \in \mathcal{U} \times H^1_D(\Omega)$ *be the solution of first-order system* (2.3). *Then it satisfies the following equations:*

$$\big(- \nabla_h \cdot (A^{\frac{1}{2}}\mathbf{u}) + Q_h(\mathbf{b} \cdot (A^{-\frac{1}{2}}\mathbf{u})) + cp, q \big) = (f, q) \quad \forall\, q \in \mathcal{S}^h_D \tag{4.3}$$

and

$$\big(\nabla_h \times (A^{-\frac{1}{2}}\mathbf{u}), r\big) = 0 \quad \forall\, r \in \mathcal{S}^h_N. \tag{4.4}$$

Proof. It follows from the definitions of the discrete divergence and curl operators and the L^2-projection and integration by parts that, for any $q \in \mathcal{S}^h_D$,

$$\big(\nabla_h \cdot (A^{\frac{1}{2}}\mathbf{u}), q\big) = \big(\nabla \cdot (A^{\frac{1}{2}}\mathbf{u}), q\big) \quad \text{and} \quad \big(Q_h(\mathbf{b} \cdot (A^{\frac{1}{2}}\mathbf{u})), q\big) = \big(\mathbf{b} \cdot (A^{\frac{1}{2}}\mathbf{u}), q\big)$$

and that, for any $r \in \mathcal{S}^h_N$,

$$\big(\nabla_h \times (A^{-\frac{1}{2}}\mathbf{u}), r\big) = \big(\nabla \times (A^{-\frac{1}{2}}\mathbf{u}), r\big) = 0,$$

which, together with the second and third equations in (2.3), imply equalities (4.3) and (4.4). ◻

For any $(\mathbf{v}, q) \in \mathcal{U}^h \times \mathcal{S}^h_D$, by (4.3) and (4.4) it is easy to see that

$$\begin{aligned} b_h(\mathbf{u}, p; \mathbf{v}, q) &= \big(f, -\nabla_h \cdot (A^{\frac{1}{2}}\mathbf{v}) + Q_h(\mathbf{b} \cdot (A^{-\frac{1}{2}}\mathbf{v})) + cq\big) \\ &\quad + \big(\nabla \cdot (A^{\frac{1}{2}}\mathbf{u}) - \nabla_h \cdot (A^{\frac{1}{2}}\mathbf{u}), cq\big) - \big((I - Q_h)(\mathbf{b} \cdot (A^{-\frac{1}{2}}\mathbf{u})), cq\big). \end{aligned} \tag{4.5}$$

The difference of equations (4.5) and (4.2) gives the following error equation:

$$\begin{aligned} &b_h(\mathbf{u} - \mathbf{u}_h, p - p_h; \mathbf{v}, q) \\ &= \big(\nabla \cdot (A^{\frac{1}{2}}\mathbf{u}) - \nabla_h \cdot (A^{\frac{1}{2}}\mathbf{u}), cq\big) - \big((I - Q_h)(\mathbf{b} \cdot (A^{-\frac{1}{2}}\mathbf{u})), cq\big) \end{aligned} \tag{4.6}$$

for all $(\mathbf{v}, q) \in \mathcal{U}^h \times \mathcal{S}^h_D$.

LEMMA 4.2. *For any* $q \in H^{\alpha-2}(\Omega)$ *with* $\alpha \geq 2$, *let* $m - 1$, *the degree of the finite element space defined in section* 3, *be the smallest integer greater than or equal to* $\alpha - 1$*; we then have that*

$$\|(I - Q_h)q\|_{H^{-1}_D(\Omega)} \leq C\, h^{\alpha-1} \|q\|_{\alpha-2}. \tag{4.7}$$

Proof. It follows from the definitions of the $H^{-1}_D(\Omega)$ norm and the L^2-projection, the Cauchy–Schwarz inequality, and the approximation property that

$$\begin{aligned} \|(I - Q_h)q\|_{H^{-1}_D(\Omega)} &= \sup_{r \in H^1_D(\Omega)} \frac{\big((I - Q_h)q, r\big)}{\|r\|_1} = \sup_{r \in H^1_D(\Omega)} \frac{\big((I - Q_h)q, (I - Q_h)r\big)}{\|r\|_1} \\ &\leq C\, h\, \|(I - Q_h)q\| \leq C\, h^{\alpha-1}\, \|q\|_{\alpha-2}. \end{aligned}$$

This completes the proof of the lemma. ◻

Now, we are ready to establish error estimates in the L^2 and H^1 norms for the vector and scalar variables, respectively, which are optimal with respect to the required regularity of the solution. Note that the norm for $\mathbf{u}$ in the error estimate in (4.8) is L^2 only but H^1 in [5]. This contributes to the less smoothness requirement of the original problem here than in [5].

THEOREM 4.1. *Assume that $(\mathbf{u}, p)$ is in $H^{\alpha-1}(\Omega)^2 \times H^\alpha(\Omega)$ with $\alpha > 1$, and let $m-1$, the degree of the finite element space defined in section 3, be the smallest integer greater than or equal to $\alpha - 1$. Then the following error estimate holds:*

$$\|\mathbf{u} - \mathbf{u}_h\| + \|p - p_h\|_1 \le C\, h^{\alpha-1}\big(\|p\|_\alpha + \|\mathbf{u}\|_{\alpha-1}\big). \tag{4.8}$$

Proof. Let p_I be an interpolant of p in $\mathcal{S}_D^h$; one then has

$$\|p_I - p\|_1 \le C\, h^{\alpha-1}\, \|p\|_\alpha. \tag{4.9}$$

To establish the error bound in (4.8), by the triangle inequality, it suffices to show that there exists a $\tilde{\mathbf{u}}_h \in \mathcal{U}^h$ such that

$$\|\mathbf{u} - \tilde{\mathbf{u}}_h\| \le C\, h^{\alpha-1}\, \|\mathbf{u}\|_{\alpha-1} \tag{4.10}$$

and that

$$\|\tilde{\mathbf{u}}_h - \mathbf{u}_h\| + \|p_I - p_h\|_1 \le C\, h^{\alpha-1}\, \big(\|p\|_\alpha + \|\mathbf{u}\|_{\alpha-1}\big). \tag{4.11}$$

Note that $\mathbf{u}$ has the decomposition of the form

$$\mathbf{u} = A^{\frac{1}{2}}\nabla s + A^{-\frac{1}{2}}\nabla^\perp t,$$

where $s \in H_D^1(\Omega)$ and $t \in H_N^1(\Omega)$ are the unique solutions of

$$\big(A\nabla s, \nabla q\big) = \big(A^{\frac{1}{2}}\mathbf{u}, \nabla q\big) \quad \forall\, q \in H_D^1(\Omega) \tag{4.12}$$

and

$$\big(A^{-1}\nabla^\perp t, \nabla^\perp r\big) = \big(A^{-\frac{1}{2}}\mathbf{u}, \nabla^\perp r\big) \quad \forall\, r \in H_N^1(\Omega), \tag{4.13}$$

respectively. Let $\tilde{s}_h \in \mathcal{S}_D^h$ and $\tilde{t}_h \in \mathcal{S}_N^h$ be the respective finite element approximations of s and t; i.e., they satisfy the following equations:

$$\big(A\nabla \tilde{s}_h, \nabla q\big) = \big(A^{\frac{1}{2}}\mathbf{u}, \nabla q\big) \quad \forall\, q \in \mathcal{S}_D^h \tag{4.14}$$

and

$$\big(A^{-1}\nabla^\perp \tilde{t}_h, \nabla^\perp r\big) = \big(A^{-\frac{1}{2}}\mathbf{u}, \nabla^\perp r\big) \quad \forall\, r \in \mathcal{S}_N^h, \tag{4.15}$$

respectively. Assume that Poisson equations (4.12) and (4.13) have the following regularity estimates:

$$\|s\|_\alpha \le C\, \|\nabla\cdot(A^{\frac{1}{2}}\mathbf{u})\|_{\alpha-2} \quad \text{and} \quad \|t\|_\alpha \le C\, \|\nabla\times(A^{-\frac{1}{2}}\mathbf{u})\|_{\alpha-2},$$

respectively. Then standard finite element error bounds give that

$$\|A^{\frac{1}{2}}\nabla(s - \tilde{s}_h)\| \le C\, h^{\alpha-1}\, \|s\|_\alpha \le C\, h^{\alpha-1}\, \|\nabla\cdot(A^{\frac{1}{2}}\mathbf{u})\|_{\alpha-2}$$

and

$$\|A^{-\frac{1}{2}}\nabla^\perp(t - \tilde{t}_h)\| \le C\, h^{\alpha-1}\, \|t\|_\alpha \le C\, h^{\alpha-1}\, \|\nabla\times(A^{-\frac{1}{2}}\mathbf{u})\|_{\alpha-2}.$$

Hence, choosing

$$\tilde{\mathbf{u}}_h = A^{\frac{1}{2}}\nabla\tilde{s}_h + A^{-\frac{1}{2}}\nabla^\perp\tilde{t}_h,$$

by orthogonality (3.2) we have that

$$\begin{aligned}\|\mathbf{u}-\tilde{\mathbf{u}}_h\| &= \Big(\|A^{\frac12}\nabla(s-\tilde{s}_h)\|^2+\|A^{-\frac12}\nabla^{\perp}(t-\tilde{t}_h)\|^2\Big)^{\frac12}\\ &\le C\,h^{\alpha-1}\,\Big(\|\nabla\cdot(A^{\frac12}\mathbf{u})\|_{\alpha-2}+\|\nabla\times(A^{-\frac12}\mathbf{u})\|_{\alpha-2}\Big).\end{aligned}$$

This completes the proof of inequality (4.10).

To show the validity of inequality (4.11), by the definition of the discrete divergence and curl operators, notice first that (4.14) and (4.15) imply

$$\nabla_h\cdot(A\nabla\tilde{s}_h)=\nabla_h\cdot(A^{\frac12}\mathbf{u})\quad\text{and}\quad\nabla_h\times(A^{-1}\nabla^{\perp}\tilde{t}_h)=\nabla_h\times(A^{-\frac12}\mathbf{u}),$$

respectively. Since $\nabla_h\cdot\nabla^{\perp}\tilde{t}_h=\nabla_h\times\nabla\tilde{s}_h=0$, we then have that

$$\nabla_h\cdot(A^{\frac12}\tilde{\mathbf{u}}_h)=\nabla_h\cdot(A^{\frac12}\mathbf{u})\quad\text{and}\quad\nabla_h\times(A^{-\frac12}\tilde{\mathbf{u}}_h)=\nabla_h\times(A^{-\frac12}\mathbf{u}).\tag{4.16}$$

It follows from Theorem 3.1 and error equation (4.6) that

$$\begin{aligned}&\frac{1}{C}|||(\tilde{\mathbf{u}}_h-\mathbf{u}_h,p_I-p_h)|||^2\\ &\le G_h(\tilde{\mathbf{u}}_h-\mathbf{u}_h,p_I-p_h;0)=b_h(\tilde{\mathbf{u}}_h-\mathbf{u}_h,p_I-p_h;\tilde{\mathbf{u}}_h-\mathbf{u}_h,p_I-p_h)\\ &=b_h(\tilde{\mathbf{u}}_h-\mathbf{u},p_I-p;\tilde{\mathbf{u}}_h-\mathbf{u}_h,p_I-p_h)-\big(\nabla\cdot(A^{\frac12}\mathbf{u})-\nabla_h\cdot(A^{\frac12}\mathbf{u}),c(p_I-p_h)\big)\\ &\quad+\big((I-Q_h)(\mathbf{b}\cdot(A^{-\frac12}\mathbf{u})),c(p_I-p_h)\big).\end{aligned}\tag{4.17}$$

Now, we bound each term in the above inequality. First, by the definitions of the discrete divergence operator, the L^2-projection, and $H_D^{-1}(\Omega)$ norm, we have that

$$\begin{aligned}&\big(\nabla\cdot(A^{\frac12}\mathbf{u})-\nabla_h\cdot(A^{\frac12}\mathbf{u}),c(p_I-p_h)\big)\\ &=\big(\nabla\cdot(A^{\frac12}\mathbf{u}),c(p_I-p_h)\big)-\big(\nabla_h\cdot(A^{\frac12}\mathbf{u}),Q_hc(p_I-p_h)\big)\\ &=\big(\nabla\cdot(A^{\frac12}\mathbf{u}),c(p_I-p_h)\big)-\big(\nabla\cdot(A^{\frac12}\mathbf{u}),Q_hc(p_I-p_h)\big)\\ &=\big((I-Q_h)\nabla\cdot(A^{\frac12}\mathbf{u}),c(p_I-p_h)\big)\\ &\le C\,\|(I-Q_h)\nabla\cdot(A^{\frac12}\mathbf{u})\|_{H_D^{-1}(\Omega)}\|p_I-p_h\|_1.\end{aligned}\tag{4.18}$$

Second, it follows from the Cauchy–Schwarz and triangle inequalities, equalities in (4.16), and the boundedness of the L^2-projection and coefficients A, $\mathbf{b}$, and c that

$$\begin{aligned}&b_h(\tilde{\mathbf{u}}_h-\mathbf{u},p_I-p;\tilde{\mathbf{u}}_h-\mathbf{u}_h,p_I-p_h)\\ &\le\Big(\|\tilde{\mathbf{u}}_h-\mathbf{u}\|+\|A^{\frac12}\nabla(p_I-p)\|\Big)\Big(\|\tilde{\mathbf{u}}_h-\mathbf{u}_h\|+\|A^{\frac12}\nabla(p_I-p_h)\|\Big)\\ &\quad+\Big(\|Q_h\big(\mathbf{b}\cdot A^{-\frac12}(\tilde{\mathbf{u}}_h-\mathbf{u})\big)\|+\|c(p_I-p)\|\Big)\cdot\\ &\qquad\Big(\|\nabla_h\cdot(A^{\frac12}(\tilde{\mathbf{u}}_h-\mathbf{u}_h))\|+\|Q_h\big(\mathbf{b}\cdot A^{-\frac12}(\tilde{\mathbf{u}}_h-\mathbf{u}_h)\big)\|+\|c(p_I-p_h)\|\Big)\\ &\le C\,\left(\|\tilde{\mathbf{u}}_h-\mathbf{u}\|+\|p_I-p\|_1\right)|||(\tilde{\mathbf{u}}_h-\mathbf{u}_h,p_I-p_h)|||.\end{aligned}\tag{4.19}$$

Substituting (4.18) and (4.19) into (4.17) implies that

$$\begin{aligned}\|\tilde{\mathbf{u}}_h-\mathbf{u}_h\|+\|p_I-p_h\|_1\le C\Big(&\|\tilde{\mathbf{u}}_h-\mathbf{u}\|+\|p_I-p\|_1+\|(I-Q_h)\big(\nabla\cdot(A^{\frac12}\mathbf{u})\big)\|_{H_D^{-1}(\Omega)}\\ &+\|(I-Q_h)\big(\mathbf{b}\cdot(A^{-\frac12}\mathbf{u})\big)\|_{H_D^{-1}(\Omega)}\Big).\end{aligned}$$

Now, (4.11) is an immediate consequence of (4.9), (4.10), and Lemma 4.2. This completes the proof of the theorem. ∎

5. Preconditioners. In this section, we discuss a spectrally equivalent preconditioner for the system of linear equations arising from the FOSLS discretization which is uniform in the mesh size.

The equivalence in Theorem 3.1 do not give us an immediate preconditioner since $|||(\mathbf{v},p)|||^2$ involves the (discrete) divergence and curl operators. Instead of working with $\mathbf{v}\in\mathcal{U}^h$, we explicitly make use of its representation:

$$\mathbf{v} = A^{\frac{1}{2}}\nabla s + A^{-\frac{1}{2}}\nabla^\perp t, \quad \text{where } s\in\mathcal{S}_D^h,\ t\in\mathcal{S}_N^h. \tag{5.1}$$

Now, $|||(\mathbf{v},p)|||^2 = |||(s,t,p)|||^2$ would be equivalent to some weighted Sobolev norm in terms of (s,t,q) which gives indications on how to construct preconditioners. To this end, by the definitions of the discrete divergence and curl operators, we first note that

$$\nabla_h\cdot(\nabla^\perp t) = 0 \quad \text{in } \Omega \quad \text{and} \quad \nabla_h\times(\nabla s) = 0 \quad \text{in } \Omega \tag{5.2}$$

for any $t\in\mathcal{S}_N^h$ and any $s\in\mathcal{S}_D^h$, respectively. We then introduce two discrete diffusion operators, $\Delta_{h,A}:\mathcal{S}_D^h\longrightarrow\mathcal{S}_D^h$ and $\hat{\Delta}_{h,A}:\mathcal{S}_N^h\longrightarrow\mathcal{S}_N^h$. For a given $s\in\mathcal{S}_D^h$, define $\Delta_{h,A}s\in\mathcal{S}_D^h$ to be the solution of

$$(\Delta_{h,A}s,\, q) = -(A\nabla s,\, \nabla q) \quad \forall\, q\in\mathcal{S}_D^h, \tag{5.3}$$

and for a given $t\in\mathcal{S}_D^h$, define $\hat{\Delta}_{h,A}t\in\mathcal{S}_N^h$ to be the solution of

$$(\hat{\Delta}_{h,A}t,\, q) = (A^{-1}\nabla^\perp t,\, \nabla^\perp q) \quad \forall\, q\in\mathcal{S}_N^h. \tag{5.4}$$

It is easy to see that

$$\Delta_{h,A} = \nabla_h\cdot A\nabla \quad \text{and} \quad \hat{\Delta}_{h,A} = \nabla_h\times A^{-1}\nabla^\perp.$$

By using (3.2), we then have that

$$|||(\mathbf{v},p)|||^2 = |||(s,t,p)|||^2 = \|p\|_1^2 + |||s|||^2 + |||t|||^2, \tag{5.5}$$

where

$$|||s|||^2 = \|s\|^2 + \|A^{\frac{1}{2}}\nabla s\|^2 + \|\Delta_{h,A}s\|^2 \quad \text{and} \quad |||t|||^2 = \|t\|^2 + \|A^{-\frac{1}{2}}\nabla^\perp t\|^2 + \|\hat{\Delta}_{h,A}t\|^2.$$

Before discussing the preconditioner based on $|||(s,t,q)|||^2$, we restate our discrete FOSLS approach in terms of functions (s,t,q). Our FOSLS functional is as follows:

$$\begin{aligned} G_h(s,t,q;f) =& \|A^{\frac{1}{2}}\nabla s + A^{-\frac{1}{2}}\nabla^\perp t - A^{\frac{1}{2}}\nabla q\|^2 \\ &+ \|f + \Delta_{h,A}s - Q_h\big(\mathbf{b}\cdot(\nabla s + A^{-1}\nabla^\perp t)\big) - cq\|^2 + \|\hat{\Delta}_{h,A}t\|^2, \end{aligned} \tag{5.6}$$

and the FOSLS minimization problem is to find $(\phi_h,\psi_h,p_h)\in\mathcal{S}_D^h\times\mathcal{S}_N^h\times\mathcal{S}_D^h$ such that

$$G_h(\phi_h,\psi_h,p_h;f) = \inf_{(s,t,q)\in\mathcal{S}_D^h\times\mathcal{S}_N^h\times\mathcal{S}_D^h} G_h(s,t,q;f) \tag{5.7}$$

with $\mathbf{u}_h = A^{\frac{1}{2}}\nabla\phi_h + A^{-\frac{1}{2}}\nabla^\perp\psi_h$. The corresponding variational problem is to find $(\phi_h,\psi_h,p_h)\in\mathcal{S}_D^h\times\mathcal{S}_N^h\times\mathcal{S}_D^h$ such that

$$b_h(\phi_h,\psi_h,p_h;s,t,q) = f_h(s,t,q) \quad \forall\,(s,t,q)\in\mathcal{S}_D^h\times\mathcal{S}_N^h\times\mathcal{S}_D^h, \tag{5.8}$$

where the bilinear and linear forms are given by

$$\begin{aligned} &b_h(\phi_h, \psi_h, p_h; s, t, q) \\ &= \big(A^{\frac12}\nabla\phi_h + A^{-\frac12}\nabla^\perp\psi_h - A^{\frac12}\nabla p_h,\ A^{\frac12}\nabla s + A^{-\frac12}\nabla^\perp t - A^{\frac12}\nabla q\big) + \big(\hat\Delta_{h,A}\psi_h,\ \hat\Delta_{h,A}t\big) \\ &+ \big(\Delta_{h,A}\phi_h - Q_h\big(\mathbf{b}\cdot(\nabla\phi_h + A^{-1}\nabla^\perp\psi_h)\big) - cp,\ \Delta_{h,A}s - Q_h\big(\mathbf{b}\cdot(\nabla s + A^{-1}\nabla^\perp t)\big) - cq\big) \end{aligned}$$

and

$$f_h(s,t,q) = \big(f,\ -\Delta_{h,A}s + Q_h\big(\mathbf{b}\cdot(\nabla s + A^{-1}\nabla^\perp t)\big) + cq\big).$$

THEOREM 5.1. *For any $(s,t,q) \in \mathcal{S}_D^h \times \mathcal{S}_N^h \times \mathcal{S}_D^h$, there exists a positive constant C such that*

$$\frac{1}{C}\,|||(s,t,q)|||^2 \le G_h(s,t,q;0) = b_h(s,t,q;s,t,q) \le C\,|||(s,t,q)|||^2. \tag{5.9}$$

Proof. It is a direct consequence of Theorem 3.1 and equality (5.5). ∎

Theorem 5.1 indicates that the quadratic form $b_h(s,t,q;s,t,q)$ can be preconditioned well by the diagonal quadratic form $|||(s,t,q)|||^2$ because they are spectrally equivalent uniformly in the mesh size (see (5.9)). We further replace these diagonal blocks of $|||(s,t,q)|||^2$ by some multigrid preconditioners. To this end, note first that $\|q\|_1^2$ is uniformly equivalent to

$$\|q\|^2 + \|A^{\frac12}\nabla q\|^2 = \big((I - \Delta_{h,A})q,\ q\big)$$

by using (2.2) and the definitions of the discrete divergence and diffusion operators. Similarly, $|||s|||^2$ and $|||t|||^2$ are uniformly equivalent to

$$\begin{aligned} &\|s\|^2 + 2\|A^{\frac12}\nabla s\|^2 + \|\Delta_{h,A}s\|^2 = \big((I - \Delta_{h,A})^2 s,\ s\big) \\ \text{and}\quad &\|t\|^2 + 2\|A^{-\frac12}\nabla^\perp s\|^2 + \|\hat\Delta_{h,A}t\|^2 = \big((I - \hat\Delta_{h,A})^2 t,\ t\big), \end{aligned}$$

respectively. Let P_1 be a preconditioner based on a symmetric multigrid V-cycle applied to the diffusion problem: find $v \in \mathcal{S}_D^h$ such that

$$(A\nabla v,\ \nabla\xi) + (v,\ \xi) = 0 \quad \forall\ \xi \in \mathcal{S}_D^h.$$

It is well known that P_1 is spectrally equivalent to $I - \Delta_{h,A}$ uniformly in the mesh size. Since the solution of

$$(I - \Delta_{h,A})^2 s = g$$

for a given $g \in \mathcal{S}_D^h$ can be obtained successively by solving two discrete diffusion equations, i.e.,

$$(I - \Delta_{h,A})\hat s = g \quad\text{and}\quad (I - \Delta_{h,A})s = \hat s,$$

it is then natural to precondition $(I - \Delta_{h,A})^2$ by P_1^2. For further discussions and numerical experiments on P_1^2 as a preconditioner for $(I - \Delta_{h,A})^2$, see [3]. Similarly, we precondition $(I - \hat\Delta_{h,A})^2$ by P_2^2, where P_2 is a preconditioner based on a symmetric multigrid V-cycle applied to the diffusion problem: find $v \in \mathcal{S}_N^h$ such that

$$(A^{-1}\nabla^\perp v,\ \nabla^\perp\xi) + (v,\ \xi) = 0 \quad \forall\ \xi \in \mathcal{S}_N^h.$$

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