

## LEAST-SQUARES METHODS FOR LINEAR ELASTICITY\*

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**Abstract.** This paper develops least-squares methods for the solution of linear elastic problems in both two and three dimensions. Our main approach is defined by simply applying the  $L^2$  norm least-squares principle to a stress-displacement system: the constitutive and the equilibrium equations. It is shown that the homogeneous least-squares functional is elliptic and continuous in the  $H(\operatorname{div}; \Omega)^d \times H^1(\Omega)^d$  norm. This immediately implies optimal error estimates for finite element subspaces of  $H(\operatorname{div}; \Omega)^d \times H^1(\Omega)^d$ . It admits optimal multigrid solution methods as well if Raviart–Thomas finite element spaces are used to approximate the stress tensor. Our method does not degrade when the material properties approach the incompressible limit. Least-squares methods that impose boundary conditions weakly and use an inverse norm are also considered. Numerical results for a benchmark test problem of planar elasticity are included in order to illustrate the robustness of our method in the incompressible limit.

**Key words.** least-squares method, mixed finite element method, linear elasticity, incompressible limit

**AMS subject classifications.** 65M60, 65M15

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**1. Introduction.** The primitive physical equations for linear elastic problems are the constitutive equation, which expresses a relation between the stress and strain tensors, and the equilibrium equation. This first-order partial differential system is called the stress-displacement formulation. Substituting the stress into the equilibrium equation leads to a second-order elliptic partial differential system called the pure displacement formulation. However, the stress-displacement formulation is preferable to the pure displacement formulation for some important practical problems, e.g., modeling of nearly incompressible or incompressible materials and modeling of plastic materials where the elimination of the stress tensor is difficult. In addition, the stress is usually a physical quantity of primary interest. It can be obtained in the pure displacement method by differentiating displacement, but this degrades the order of the approximation.

A mixed finite element method is based on the weak form of the stress-displacement formulation, and it requires a stable combination of finite element spaces to approximate these variables. Unlike mixed methods for second-order scalar elliptic boundary value problems, stress-displacement finite elements are extremely difficult to construct. This is caused by the symmetry constraint of the stress tensor. Recently, Arnold and Winther in [3] constructed a family of stable conforming elements in two dimensions on a triangular tessellation. Their simplest element has 21 stress and 3 displacement degrees of freedom per triangle. The local degrees of freedom are reduced to 12 for the stress and 3 for the displacement for a stable nonconforming element in [4]. For previous work on mixed methods for linear elasticity, see [3] and

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references therein. Like scalar elliptic problems, mixed methods lead to saddle-point problems. Many solution methods that work well for symmetric and positive definite problems cannot be applied directly. Although substantial progress in solution methods for saddle-point problems has been achieved, these problems may still be difficult and expensive to solve.

In recent years there has been increasing interest in the use of least-squares principles for numerical approximations of partial differential equations and systems (see, e.g., the survey paper [6], the monograph [18], and references therein). Their advantages over the usual mixed finite element discretizations include that the choice of finite element spaces is not subject to the stability condition (see, e.g., [9]), that the resulting algebraic equations can be solved efficiently by standard multigrid methods or preconditioned by well-known techniques, and that the value of a least-squares functional provides a free, sharp, and practical a posteriori error indicator which can be used efficiently in a local refinement process. For linear elasticity, in particular, least-squares methods have an additional edge over mixed methods in that the known stable mixed elements are very limited and they have a large number of degrees of freedom. In [11], Cai, Manteuffel, McCormick, and Parter proposed a two-stage least-squares approach that first solves for the displacement gradient and then solves for the displacement itself (if desired). Physical quantities such as the strain, the stress, and the rotation are then simple linear combinations of the displacement gradient. At the first stage, it has four (nine) variables in two (three) dimensions, compared to five (nine) variables for the stress-displacement formulation. One drawback of this approach is its requirement of sufficient smoothness on the original problem if using standard continuous finite element approximations. Another approach was proposed by engineers in [19] based on a displacement-stress-rotation formulation; it has the same drawback as that of [11]. In addition, it introduces extra variables (the rotation): one (three) variable in two (three) dimensions. For other least-squares approaches in the engineering literature in solid mechanics, see references in [19].

In contrast to these approaches, our aim is to develop a least-squares approach that does not have the above-mentioned drawbacks, and that computes the stress and the displacement directly. Thus it would be easier to extend this method to applications such as nonlinear elasticity, plasticity, etc. The stress components are physical quantities of primary interest in many practical applications including coupling of elastic deformation with fluid flow models. The method to be developed in this paper is based on the primitive physical equations of linear elasticity: the stress-displacement formulation, *without introducing any new variables or any new equations*. Applying the  $L^2$  norm least-squares principle to this first-order system with an appropriately scaled constitutive equation, we develop a least-squares formulation for linear elasticity. It is shown that the homogeneous least-squares functional is elliptic and continuous in the  $H(\text{div}; \Omega)$  norm for the stress and in the  $H^1$  norm for the displacement uniformly with respect to material constants. This immediately implies optimal error estimates for finite element subspaces of  $H(\text{div}; \Omega)^d \times H^1(\Omega)^d$ . It also admits optimal multigrid solution methods if Raviart–Thomas finite element spaces (see, e.g., [9]) are used to approximate the stress tensor. Both discretization accuracy and multigrid convergence rate of the method do not degrade when the material properties approach the incompressible limit. As usual, the evaluation of the least-squares functional on each element is a practical and sharp a posteriori error indicator for adaptive mesh refinements. The practical performance of the resulting

adaptive strategy will be tested numerically for a common benchmark problem of linear elasticity in the final section of this paper.

The method here is closely related to our previous work in [12, 10]. The main difference is the scale in the constitutive equation. The homogeneous least-squares functionals in [12, 10] are equivalent to the  $H(\operatorname{div}; \Omega)$  norm for the stress and the *energy* norm for the displacement. This means that the least-squares variational problems in [12, 10] do not apply for incompressible materials and require effective discretizations and efficient solvers for the pure displacement problem when materials are nearly incompressible. These tasks remain difficult and expensive although some progress has been achieved (see, e.g., [8] and references therein).

For completeness, we study an inverse norm least-squares functional and show that its homogeneous form is equivalent to the  $L^2(\Omega)^{d \times d} \times H^1(\Omega)^d$  norm for the stress and the displacement. This functional can be used to develop a discrete inverse norm least-squares method (see, e.g., [7]). For some applications, it is convenient to impose boundary conditions weakly by adding boundary functionals. Such a functional is also studied in this paper. See [21] for how to use these types of functionals to develop a computationally feasible numerical method.

An outline of the paper is as follows. The stress-displacement formulation for the linear elastic problem is introduced in section 2, along with some notation, the pure displacement formulation, and some regularity estimates. Section 3 develops the least-squares functionals based on the stress-displacement formulation and establishes their ellipticity and continuity. Section 4 discusses finite element approximations. Section 5 studies a least-squares functional with boundary terms that enforces boundary conditions weakly. Finally, numerical results for a benchmark test problem of linear elasticity are presented in section 6.

**1.1. Notation.** We use the standard notation and definitions for the Sobolev spaces  $H^s(\Omega)^d$  and  $H^s(\partial\Omega)^d$  for  $s \geq 0$ ; the standard associated inner products are denoted by  $(\cdot, \cdot)_{s, \Omega}$  and  $(\cdot, \cdot)_{s, \partial\Omega}$ , and their respective norms are denoted by  $\|\cdot\|_{s, \Omega}$  and  $\|\cdot\|_{s, \partial\Omega}$ . (We suppress the superscript  $d$  because their dependence on dimension will be clear by context. We also omit the subscript  $\Omega$  from the inner product and norm designation when there is no risk of confusion.) For  $s = 0$ ,  $H^s(\Omega)^d$  coincides with  $L^2(\Omega)^d$ . In this case, the inner product and norm will be denoted by  $\|\cdot\|$  and  $(\cdot, \cdot)$ , respectively. Set  $H_D^1(\Omega) := \{q \in H^1(\Omega) : q = 0 \text{ on } \Gamma_D\}$ . We use  $H_D^{-1}(\Omega)$  and  $H^{-\frac{1}{2}}(\partial\Omega)$  to denote the dual of  $H_D^1(\Omega)$  and  $H^{\frac{1}{2}}(\partial\Omega)$  with norms defined by

$$\|\phi\|_{-1, D} = \sup_{0 \neq \psi \in H_D^1(\Omega)} \frac{(\phi, \psi)}{\|\psi\|_1} \quad \text{and} \quad \|\phi\|_{-1/2, \partial\Omega} = \sup_{0 \neq \psi \in H^{\frac{1}{2}}(\partial\Omega)} \frac{(\phi, \psi)}{\|\psi\|_{1/2, \partial\Omega}}.$$

Denote the product space  $H_D^{-1}(\Omega)^d = \prod_{i=1}^d H_D^{-1}(\Omega)$  with the standard product norm. Set

$$H(\operatorname{div}; \Omega) = \{\mathbf{v} \in L^2(\Omega)^2 : \nabla \cdot \mathbf{v} \in L^2(\Omega)\},$$

which is a Hilbert space under the norm

$$\|\mathbf{v}\|_{H(\operatorname{div}; \Omega)} = (\|\mathbf{v}\|^2 + \|\nabla \cdot \mathbf{v}\|^2)^{\frac{1}{2}}.$$

**2. Linear elasticity and preliminaries.** Let  $\Omega$  be a bounded, open, connected subset of  $\mathbb{R}^d$  ( $d = 2$  or  $3$ ) with a Lipschitz continuous boundary  $\partial\Omega$ . Denote  $\mathbf{n} = (n_1, \dots, n_d)^t$  as the outward unit vector normal to the boundary. We partition the

boundary of the domain  $\Omega$  into two open subsets  $\Gamma_D$  and  $\Gamma_N$  such that  $\partial\Omega = \bar{\Gamma}_D \cup \bar{\Gamma}_N$  and  $\Gamma_D \cap \Gamma_N = \emptyset$ . For simplicity, we assume that  $\Gamma_D$  is not empty (i.e.,  $\text{mes}(\Gamma_D) \neq 0$ ). Our approaches proposed in this paper can be easily extended to the pure traction problem ( $\Gamma_D = \emptyset$ ) by excluding the space of infinitesimal rigid motions.

Let  $\mathbf{f} = (f_1, \dots, f_d)^t$  be a given body force defined on  $\Omega$ . The linear elastic problem consists of finding a displacement field  $\mathbf{u} = (u_1, \dots, u_d)^t$  and a stress tensor  $\boldsymbol{\sigma} = (\sigma_{ij})_{d \times d}$  that satisfy the equilibrium equation

$$(2.1) \quad \sum_{j=1}^d \frac{\partial \sigma_{ij}}{\partial x_j} + f_i = 0 \quad \text{for } i = 1, \dots, d$$

and boundary conditions

$$(2.2) \quad \mathbf{u} = \mathbf{0} \quad \text{on } \Gamma_D \quad \text{and} \quad \sum_{j=1}^d \sigma_{ij} n_j = 0 \quad \text{on } \Gamma_N \quad \text{for } i = 1, \dots, d.$$

For simplicity, here we assume that the boundary conditions are homogeneous.

Denote  $\boldsymbol{\epsilon}(\mathbf{u}) = (\epsilon_{ij}(\mathbf{u}))_{d \times d}$  as the linearized strain tensor, where

$$\epsilon_{ij}(\mathbf{u}) = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right).$$

The constitutive law expresses a relation between the stress and the strain tensors:

$$(2.3) \quad \boldsymbol{\sigma} = \mathcal{C} \boldsymbol{\epsilon}(\mathbf{u}) \quad \text{or} \quad \boldsymbol{\epsilon}(\mathbf{u}) = \mathcal{A} \boldsymbol{\sigma},$$

where  $\mathcal{C}$  and  $\mathcal{A}$  are the elasticity and the compliance tensors of fourth order, respectively. Denote by  $\text{tr}$  the trace operator

$$\text{tr}(\boldsymbol{\epsilon}(\mathbf{u})) = \epsilon_{11}(\mathbf{u}) + \dots + \epsilon_{dd}(\mathbf{u}) = \nabla \cdot \mathbf{u},$$

where  $\nabla \cdot$  stands for the divergence operator. For an isotropic elastic material, the elasticity tensor has the following simple expression:

$$(2.4) \quad \mathcal{C} \boldsymbol{\epsilon}(\mathbf{u}) = \lambda \text{tr}(\boldsymbol{\epsilon}(\mathbf{u})) \boldsymbol{\delta} + 2\mu \boldsymbol{\epsilon}(\mathbf{u}),$$

where  $\boldsymbol{\delta} = (\delta_{ij})_{d \times d}$  is the identity tensor, and positive constants  $\lambda$  and  $\mu$  are the Lamé constants such that  $\mu \in [\mu_1, \mu_2]$  with  $0 < \mu_1 < \mu_2$  and  $\lambda \in (0, \infty)$ . Materials are said to be nearly incompressible or incompressible when  $\lambda$  is very large or infinite, respectively. Note that both the stress and the strain tensors are symmetric. Such symmetry of the stress stems from the conservation of angular momentum.

For a second-order tensor  $\boldsymbol{\tau} = (\tau_{ij})_{d \times d}$ , define its divergence and normal by

$$\nabla \cdot \boldsymbol{\tau} = \begin{pmatrix} \partial \tau_{11} / \partial x_1 + \dots + \partial \tau_{1d} / \partial x_d \\ \vdots \\ \partial \tau_{d1} / \partial x_1 + \dots + \partial \tau_{dd} / \partial x_d \end{pmatrix} \quad \text{and} \quad \mathbf{n} \cdot \boldsymbol{\tau} = \begin{pmatrix} n_1 \tau_{11} + \dots + n_d \tau_{1d} \\ \vdots \\ n_1 \tau_{d1} + \dots + n_d \tau_{dd} \end{pmatrix},$$

respectively. That is, the divergence and normal operators apply to each row of the tensor. Then the stress-displacement system in (2.3), (2.1), and (2.2) may be rewritten in the compact form

$$(2.5) \quad \begin{cases} \boldsymbol{\sigma} - \mathcal{C} \boldsymbol{\epsilon}(\mathbf{u}) & = \mathbf{0} & \text{in } \Omega, \\ \nabla \cdot \boldsymbol{\sigma} & = -\mathbf{f} & \text{in } \Omega \end{cases}$$

with the boundary conditions

$$(2.6) \quad \mathbf{u} = \mathbf{0} \quad \text{on } \Gamma_D \quad \text{and} \quad \mathbf{n} \cdot \boldsymbol{\sigma} = \mathbf{0} \quad \text{on } \Gamma_N.$$

There are three approaches to treating this first-order partial differential system. One is to substitute the stress into the equilibrium equation to get the pure displacement formulation in (2.7). Numerical methods based on this formulation are not desirable for accurate approximations of the stress and for some important practical problems such as the modeling of nearly incompressible or incompressible or plastic materials. Another approach is to find the unique saddle point  $(\boldsymbol{\sigma}, \mathbf{u}) \in H_N^S(\text{div}; \Omega)^d \times L^2(\Omega)^d$  of the Hellinger–Reissner functional

$$\mathcal{J}(\boldsymbol{\tau}, \mathbf{v}) = \frac{1}{2}(\mathcal{A}\boldsymbol{\tau}, \boldsymbol{\tau}) + (\nabla \cdot \boldsymbol{\tau} + \mathbf{f}, \mathbf{v}).$$

Here  $H_N^S(\text{div}; \Omega)^d$  denotes the space of square-integrable symmetric tensors with square-integrable divergence and homogeneous normal on  $\Gamma_N$ . Equivalently,  $(\boldsymbol{\sigma}, \mathbf{u})$  satisfies the following weak formulation:

$$(\mathcal{A}\boldsymbol{\sigma}, \boldsymbol{\tau}) + (\mathbf{u}, \nabla \cdot \boldsymbol{\tau}) + (\nabla \cdot \boldsymbol{\sigma}, \mathbf{v}) = (-\mathbf{f}, \mathbf{v}) \quad \forall (\boldsymbol{\tau}, \mathbf{v}) \in H_N^S(\text{div}; \Omega)^d \times L^2(\Omega)^d.$$

Numerical methods based on this formulation require a stable combination of finite element spaces to approximate the stress and the displacement. Known stable mixed elements are very limited and have a large number of degrees of freedom. In addition, the resulting indefinite algebraic system is still difficult and expensive to solve. In this paper, we study the third approach based on the least-squares principle that automatically stabilizes the stress-displacement system (see section 3).

We complete this section by deriving the pure displacement formulation and describing some regularity estimates. To this end, eliminating the stress in system (2.5)–(2.6) yields the pure displacement formulation which satisfies the following second-order elliptic partial differential system:

$$(2.7) \quad \begin{cases} 2\mu \nabla \cdot \boldsymbol{\epsilon}(\mathbf{u}) + \lambda \nabla(\nabla \cdot \mathbf{u}) &= -\mathbf{f} & \text{in } \Omega, \\ \mathbf{u} &= \mathbf{0} & \text{on } \Gamma_D, \\ \mathbf{n} \cdot (2\mu \boldsymbol{\epsilon}(\mathbf{u}) + \lambda(\nabla \cdot \mathbf{u})\boldsymbol{\delta}) &= \mathbf{0} & \text{on } \Gamma_N, \end{cases}$$

where  $\nabla$  stands for the gradient operator. The energy norm associated with the above problem is defined as follows:

$$(2.8) \quad |||\mathbf{v}||| = (2\mu \|\boldsymbol{\epsilon}(\mathbf{v})\|^2 + \lambda \|\nabla \cdot \mathbf{v}\|^2)^{\frac{1}{2}}.$$

By using Korn’s inequality (see [14]),

$$(2.9) \quad \|\mathbf{v}\|_1 \leq C \|\boldsymbol{\epsilon}(\mathbf{v})\| \quad \forall \mathbf{v} \in H_D^1(\Omega)^d,$$

the energy norm is equivalent to the  $H^1$  norm for a fixed  $\lambda$ . In this paper, we use  $C$  with or without subscripts to denote a generic positive constant, possibly different at different occurrences, which is independent of the Lamé constant  $\lambda$  and the mesh size  $h$  introduced in section 4 but may depend on the domain  $\Omega$ . Note that one could scale the variables and the right-hand side accordingly so that  $\mu$  is equal to one. We will frequently use the term *uniform* in reference to a relation to mean that it holds independent of  $\lambda$  and  $h$ .

The weak form of boundary value problem (2.7) has a unique solution  $\mathbf{u} \in H_D^1(\Omega)^d$  for any  $\mathbf{f} \in H_D^{-1}(\Omega)^d$  (see [14]). Moreover, the solution satisfies the following  $H^1$ -regularity estimate (see, e.g., [8, 14]):

$$(2.10) \quad \|\mathbf{u}\|_1 + \lambda \|\nabla \cdot \mathbf{u}\| \leq C \|\mathbf{f}\|_{-1,D}.$$

Furthermore, if the domain  $\Omega$  is convex or its boundary is  $C^{1,1}$  and if either  $\Gamma_D$  or  $\Gamma_N$  is empty, then the  $H^2$ -regularity estimate

$$(2.11) \quad \|\mathbf{u}\|_2 + \lambda \|\nabla \cdot \mathbf{u}\|_1 \leq C \|\mathbf{f}\|$$

holds. Both the regularity estimates in (2.10) and (2.11) suggest that the divergence of the displacement has a different scale from the displacement itself for large  $\lambda$ .

**3. Least-squares variational formulation.** In this section, we first discuss an appropriate stress-displacement formulation and then consider the corresponding least-squares functionals based on such a first-order system. Our primary objective here is to establish continuity and ellipticity of these least-squares functionals in the appropriate Hilbert spaces.

It is convenient to view  $d \times d$ -matrices as  $d^2$ -vectors and vice versa. For example, view  $(\sigma_{ij})_{d \times d}$  as  $(\boldsymbol{\sigma}_1, \dots, \boldsymbol{\sigma}_d)^t$ , where  $\boldsymbol{\sigma}_j = (\sigma_{j1}, \dots, \sigma_{jd})$  is the  $j$ th row of  $(\sigma_{ij})_{d \times d}$  for  $j = 1, \dots, d$ . Let

$$\mathbf{b} = \begin{cases} (1, 0, 0, 1)^t & d = 2, \\ (1, 0, 0, 0, 1, 0, 0, 0, 1)^t & d = 3, \end{cases}$$

which may be viewed as the  $d \times d$  identity matrix  $I_{d \times d}$  or the identity tensor  $\boldsymbol{\delta}$ . Thus,

$$\text{tr } \boldsymbol{\sigma} = \text{tr } (\sigma_{ij})_{d \times d} = \sum_{i=1}^d \sigma_{ii} = \mathbf{b}^t \begin{pmatrix} \boldsymbol{\sigma}_1^t \\ \vdots \\ \boldsymbol{\sigma}_d^t \end{pmatrix} = \mathbf{b}^t \boldsymbol{\sigma}.$$

By viewing a  $d \times d$ -matrix as a  $d^2$ -vector, we can then write the fourth-order elasticity tensor as  $d^2 \times d^2$ -matrix

$$\mathcal{C} = \lambda \mathbf{b} \mathbf{b}^t + 2\mu I.$$

It is clear that  $\mathcal{C}$  is symmetric and that  $\mathcal{C}$  is positive definite for finite  $\lambda$ . The compliance tensor has the form

$$(3.1) \quad \mathcal{A} = \frac{1}{2\mu} \left( I - \frac{\lambda}{d\lambda + 2\mu} \mathbf{b} \mathbf{b}^t \right).$$

Note that  $\mathcal{A} = \mathcal{C}^{-1}$  for finite  $\lambda$ . When the  $\lambda$  approaches  $\infty$ , the elasticity tensor blows up and the compliance tensor tends to

$$(3.2) \quad \frac{1}{2\mu} \left( I - \frac{1}{d} \mathbf{b} \mathbf{b}^t \right) \equiv \frac{1}{2\mu} \mathbf{dev},$$

which is not invertible. For any tensor  $\boldsymbol{\tau}$ ,  $\mathbf{dev} \boldsymbol{\tau} = \boldsymbol{\tau} - \frac{1}{d} (\text{tr } \boldsymbol{\tau}) \boldsymbol{\delta}$  is the deviatoric part of  $\boldsymbol{\tau}$ . Hence, for nearly incompressible or incompressible materials, it is preferable to use the following strain and stress relation:

$$(3.3) \quad \boldsymbol{\epsilon}(\mathbf{u}) = \mathcal{A} \boldsymbol{\sigma} = \frac{1}{2\mu} \left( \boldsymbol{\sigma} - \frac{\lambda}{d\lambda + 2\mu} (\text{tr } \boldsymbol{\sigma}) \boldsymbol{\delta} \right).$$

Now, the first-order system for the stress and the displacement of linear elasticity is as follows:

$$(3.4) \quad \begin{cases} \mathcal{A}\boldsymbol{\sigma} - \boldsymbol{\epsilon}(\mathbf{u}) = \mathbf{0} & \text{in } \Omega, \\ \nabla \cdot \boldsymbol{\sigma} + \mathbf{f} = \mathbf{0} & \text{in } \Omega \end{cases}$$

with boundary conditions (2.6), where  $\mathcal{A}$  is given in (3.1).

It is important to note that the stress is symmetric; i.e.,

$$(3.5) \quad \boldsymbol{\sigma} - \boldsymbol{\sigma}^t = \mathbf{0},$$

where  $\boldsymbol{\sigma}^t$  denotes the transpose of  $\boldsymbol{\sigma}$  as a  $d \times d$ -matrix. One can impose this symmetry condition either in the solution space (in a strong sense) or in the equation (in a weak sense). In [12], we augment (3.5) with the stress-displacement system so that our least-squares methods have freedom to treat it either strongly or weakly depending on discretization and solution methods. In [10], we show that the symmetry constraint of the stress is enforced at the continuous level even without the term  $\|\boldsymbol{\sigma} - \boldsymbol{\sigma}^t\|^2$  in the least-squares functionals. This is because for any  $\boldsymbol{\tau} \in L^2(\Omega)^{d \times d}$  and any  $\mathbf{v} \in H^1(\Omega)^d$ , we have

$$(3.6) \quad \|\boldsymbol{\tau} - \boldsymbol{\tau}^t\| \leq C \|\mathcal{A}\boldsymbol{\tau} - \boldsymbol{\epsilon}(\mathbf{v})\|.$$

Thus,  $\mathcal{A}\boldsymbol{\tau} - \boldsymbol{\epsilon}(\mathbf{v}) = 0$  implies that  $\boldsymbol{\tau}$  is symmetric. Therefore, we will apply the least-squares principle to first-order system (3.4) without augmenting (3.5). Inequality (3.6) follows from the symmetry of  $\boldsymbol{\epsilon}(\mathbf{v})$ , (3.1), and the triangle inequality that

$$\begin{aligned} \|\boldsymbol{\tau} - \boldsymbol{\tau}^t\| &= 2\mu \left\| \left( \frac{1}{2\mu} \boldsymbol{\tau} - \boldsymbol{\epsilon}(\mathbf{v}) \right) - \left( \frac{1}{2\mu} \boldsymbol{\tau} - \boldsymbol{\epsilon}(\mathbf{v}) \right)^t \right\| \\ &= 2\mu \| (\mathcal{A}\boldsymbol{\tau} - \boldsymbol{\epsilon}(\mathbf{v})) - (\mathcal{A}\boldsymbol{\tau} - \boldsymbol{\epsilon}(\mathbf{v}))^t \| \leq 4\mu \|\mathcal{A}\boldsymbol{\tau} - \boldsymbol{\epsilon}(\mathbf{v})\|. \end{aligned}$$

Before defining least-squares functionals, let us first describe solution spaces. If  $\Gamma_N = \emptyset$ , then  $\int_{\Omega} \nabla \cdot \mathbf{u} \, dx = \int_{\partial\Omega} \mathbf{n} \cdot \mathbf{u} \, ds = 0$ , which implies

$$\int_{\Omega} \text{tr } \boldsymbol{\sigma} \, dx = 0.$$

Therefore, we are at liberty to impose such a condition for the stress  $\boldsymbol{\sigma}$ . Let

$$\mathbf{X} = \begin{cases} H(\text{div}; \Omega)^d & \text{if } \Gamma_N \neq \emptyset, \\ \{\boldsymbol{\tau} \in H(\text{div}; \Omega)^d \mid \int_{\Omega} \text{tr } \boldsymbol{\sigma} \, dx = 0\} & \text{otherwise,} \end{cases}$$

and denote its subspace by

$$\mathbf{X}_N = \{\boldsymbol{\tau} \in \mathbf{X} : \mathbf{n} \cdot \boldsymbol{\tau} = \mathbf{0} \text{ on } \Gamma_N\}.$$

Let

$$\mathcal{V}_B = \mathbf{X}_N \times H_D^1(\Omega)^d.$$

For  $\mathbf{f} \in L^2(\Omega)^d$ , we define the following least-squares functionals:

$$(3.7) \quad G_{-1}(\boldsymbol{\sigma}, \mathbf{u}; \mathbf{f}) = \|\mathcal{A}\boldsymbol{\sigma} - \boldsymbol{\epsilon}(\mathbf{u})\|^2 + \|\nabla \cdot \boldsymbol{\sigma} + \mathbf{f}\|_{-1,D}^2$$

and

$$(3.8) \quad G(\boldsymbol{\sigma}, \mathbf{u}; \mathbf{f}) = \|\mathcal{A}\boldsymbol{\sigma} - \boldsymbol{\epsilon}(\mathbf{u})\|^2 + \|\nabla \cdot \boldsymbol{\sigma} + \mathbf{f}\|^2$$

for  $(\boldsymbol{\sigma}, \mathbf{u}) \in \mathcal{V}_B$ . Least-squares variational problems for the stress-displacement of linear elasticity are then to minimize the above least-squares functionals over  $\mathcal{V}_B$ . In this paper, we concentrate on the least-squares problem based on the  $L^2$  norm functional in (3.8): find  $(\boldsymbol{\sigma}, \mathbf{u}) \in \mathcal{V}_B$  such that

$$(3.9) \quad G(\boldsymbol{\sigma}, \mathbf{u}; \mathbf{f}) = \inf_{(\boldsymbol{\tau}, \mathbf{v}) \in \mathcal{V}_B} G(\boldsymbol{\tau}, \mathbf{v}; \mathbf{f}).$$

Note that the inverse norm functional in (3.7) can be used to develop a discrete inverse norm least-squares method (see [7]) as well.

REMARK 3.1. *Since the minimum of the quadratic functional  $G(\boldsymbol{\sigma}, \mathbf{u}; \mathbf{f})$  is zero, by (3.6) the symmetry of the stress tensor is guaranteed by the first term of the functional, i.e., the constitutive equation.*

REMARK 3.2. *The least-squares functionals defined in (3.7) and (3.8) differ from those in [12, 10] mainly in the first term with an extra weight of  $\mathcal{C}^{-\frac{1}{2}}$ . More precisely, the first term of the functionals in [12, 10] is  $\|\mathcal{C}^{-\frac{1}{2}}\boldsymbol{\sigma} - \mathcal{C}^{\frac{1}{2}}\boldsymbol{\epsilon}(\mathbf{u})\|^2$ . Note that  $\|\mathcal{C}^{\frac{1}{2}}\boldsymbol{\epsilon}(\mathbf{u})\|^2 = \lambda \|\nabla \mathbf{u}\|^2 + 2\mu \|\boldsymbol{\epsilon}(\mathbf{u})\|^2$ . This means that the least-squares variational problems in [12, 10] do not apply for incompressible materials and require effective discretizations and efficient solvers for the pure displacement problem when materials are nearly incompressible.*

Below we establish uniform continuity and ellipticity (i.e., equivalence) of the homogeneous functionals  $G_{-1}(\boldsymbol{\tau}, \mathbf{v}; \mathbf{0})$  and  $G(\boldsymbol{\tau}, \mathbf{v}; \mathbf{0})$  in terms of the respective functionals  $M_{-1}(\boldsymbol{\tau}, \mathbf{v})$  and  $M(\boldsymbol{\tau}, \mathbf{v})$  defined on  $\mathcal{V}_B$  by

$$M_{-1}(\boldsymbol{\tau}, \mathbf{v}) = \|\boldsymbol{\epsilon}(\mathbf{v})\|^2 + \|\boldsymbol{\tau}\|^2$$

and

$$M(\boldsymbol{\tau}, \mathbf{v}) = \|\boldsymbol{\epsilon}(\mathbf{v})\|^2 + \|\boldsymbol{\tau}\|^2 + \|\nabla \cdot \boldsymbol{\tau}\|^2.$$

To do so, we need the following fundamental inequality on the trace of  $\mathbf{X}_N$ :

$$(3.10) \quad \|\text{tr } \boldsymbol{\tau}\| \leq C \left( \sqrt{(\mathcal{A}\boldsymbol{\tau}, \boldsymbol{\tau})} + \|\nabla \cdot \boldsymbol{\tau}\|_{-1,D} \right) \quad \forall \boldsymbol{\tau} \in \mathbf{X}_N,$$

where  $C$  is a positive constant independent of  $\lambda$ . This inequality should be a classic result. But the only references that we know for its proof are [1] for two dimensions and Dirichlet boundary conditions (i.e.,  $d = 2$  and  $\Gamma_N = \emptyset$ ) and [12] for both two and three dimensions and general boundary conditions. Note that

$$(3.11) \quad \begin{aligned} (\mathcal{A}\boldsymbol{\tau}, \boldsymbol{\tau}) &= \frac{1}{2\mu} \left( \|\boldsymbol{\tau}\|^2 - \frac{\lambda}{d\lambda + 2\mu} \|\text{tr } \boldsymbol{\tau}\|^2 \right) \\ &= \frac{1}{2\mu} \|\mathbf{dev } \boldsymbol{\tau}\|^2 + \frac{1}{d(d\lambda + 2\mu)} \|\text{tr } \boldsymbol{\tau}\|^2, \end{aligned}$$

where  $\mathbf{dev } \boldsymbol{\tau}$  and  $\text{tr } \boldsymbol{\tau}$  are the respective deviatoric and volumetric parts of  $\boldsymbol{\tau}$ . It is then obvious that the divergence term in (3.10) is necessary to bound the  $L^2$  norm of the trace. From the definition of the inverse norm and the Cauchy–Schwarz inequality, we have that

$$(3.12) \quad \|\nabla \cdot \boldsymbol{\tau}\|_{-1,D} \leq \|\boldsymbol{\tau}\|.$$

By (3.10), (3.11), and (3.12) it is easy to see that

$$\|\boldsymbol{\tau}\|_a \equiv \left( (\mathcal{A}\boldsymbol{\tau}, \boldsymbol{\tau}) + \|\nabla \cdot \boldsymbol{\tau}\|_{-1, D}^2 \right)^{\frac{1}{2}}$$

is equivalent to the  $L^2$  norm; i.e., there exists a positive constant  $C$  independent of  $\lambda$  such that

$$(3.13) \quad \frac{1}{C} \|\boldsymbol{\tau}\|^2 \leq \|\boldsymbol{\tau}\|_a^2 \leq C \|\boldsymbol{\tau}\|^2 \quad \forall \boldsymbol{\tau} \in \mathbf{X}_N.$$

**THEOREM 3.1.** *The homogeneous functionals  $G_{-1}(\boldsymbol{\tau}, \mathbf{v}; \mathbf{0})$  and  $G(\boldsymbol{\tau}, \mathbf{v}; \mathbf{0})$  are uniformly equivalent to the functionals  $M_{-1}(\boldsymbol{\tau}, \mathbf{v})$  and  $M(\boldsymbol{\tau}, \mathbf{v})$ , respectively; i.e., there exist positive constants  $C_1$  and  $C_2$ , independent of  $\lambda$ , such that*

$$(3.14) \quad \frac{1}{C_1} M_{-1}(\boldsymbol{\tau}, \mathbf{v}) \leq G_{-1}(\boldsymbol{\tau}, \mathbf{v}; \mathbf{0}) \leq C_1 M_{-1}(\boldsymbol{\tau}, \mathbf{v})$$

and

$$(3.15) \quad \frac{1}{C_2} M(\boldsymbol{\tau}, \mathbf{v}) \leq G(\boldsymbol{\tau}, \mathbf{v}; \mathbf{0}) \leq C_2 M(\boldsymbol{\tau}, \mathbf{v})$$

hold for all  $(\boldsymbol{\tau}, \mathbf{v}) \in \mathcal{V}_B$ .

*Proof.* It follows from (3.1) that

$$\begin{aligned} \|\mathcal{A}\boldsymbol{\tau}\|^2 &= \left( \frac{1}{2\mu} \right)^2 \left( \|\boldsymbol{\tau}\|^2 - \frac{2\lambda}{d\lambda + 2\mu} \|\text{tr } \boldsymbol{\tau}\|^2 + d \left( \frac{\lambda}{d\lambda + 2\mu} \right)^2 \|\text{tr } \boldsymbol{\tau}\|^2 \right) \\ &= \left( \frac{1}{2\mu} \right)^2 \left( \|\boldsymbol{\tau}\|^2 - \frac{\lambda(d\lambda + 4\mu)}{(d\lambda + 2\mu)^2} \|\text{tr } \boldsymbol{\tau}\|^2 \right) \leq \left( \frac{1}{2\mu} \right)^2 \|\boldsymbol{\tau}\|^2. \end{aligned}$$

Thus,  $\mathcal{A}\boldsymbol{\tau}$  is bounded above by  $\boldsymbol{\tau}$  in the  $L^2$  norm:

$$(3.16) \quad \|\mathcal{A}\boldsymbol{\tau}\| \leq \frac{1}{2\mu} \|\boldsymbol{\tau}\|.$$

The upper bounds in both (3.14) and (3.15) follow easily from the triangle inequality, (3.16), and (3.12). To show the validity of the lower bound in (3.14), we first prove that  $\boldsymbol{\tau}$  in the  $L^2$  norm is bounded above by the homogeneous functional:

$$(3.17) \quad \|\boldsymbol{\tau}\|^2 \leq C G_{-1}(\boldsymbol{\tau}, \mathbf{v}; \mathbf{0}) \quad \forall (\boldsymbol{\tau}, \mathbf{v}) \in \mathcal{V}_B.$$

To this end, by the triangle inequality and (3.16) we have

$$(3.18) \quad \|\boldsymbol{\epsilon}(\mathbf{v})\| \leq \|\boldsymbol{\epsilon}(\mathbf{v}) - \mathcal{A}\boldsymbol{\tau}\| + \|\mathcal{A}\boldsymbol{\tau}\| \leq \|\boldsymbol{\epsilon}(\mathbf{v}) - \mathcal{A}\boldsymbol{\tau}\| + \frac{1}{2\mu} \|\boldsymbol{\tau}\|.$$

Since  $\boldsymbol{\epsilon}(\mathbf{v}) = \frac{1}{2}(\nabla\mathbf{v} + (\nabla\mathbf{v})^t)$  is symmetric, then integration by parts; the triangle, Cauchy-Schwarz, and Korn inequalities; and (3.6) give

$$\begin{aligned} |(\boldsymbol{\tau}, \boldsymbol{\epsilon}(\mathbf{v}))| &= \left| (\boldsymbol{\tau}, \nabla\mathbf{v}) - \left( \frac{\boldsymbol{\tau} - \boldsymbol{\tau}^t}{2}, \nabla\mathbf{v} \right) \right| = \left| (\nabla \cdot \boldsymbol{\tau}, \mathbf{v}) + \left( \frac{\boldsymbol{\tau} - \boldsymbol{\tau}^t}{2}, \nabla\mathbf{v} \right) \right| \\ &\leq \left( \|\nabla \cdot \boldsymbol{\tau}\|_{-1, D} + \left\| \frac{\boldsymbol{\tau} - \boldsymbol{\tau}^t}{2} \right\| \right) \|\mathbf{v}\|_1 \\ &\leq C (\|\nabla \cdot \boldsymbol{\tau}\|_{-1, D} + \|\mathcal{A}\boldsymbol{\tau} - \boldsymbol{\epsilon}(\mathbf{v})\|) \|\boldsymbol{\epsilon}(\mathbf{v})\|, \end{aligned}$$

which, together with (3.18), implies that

$$(3.19) \quad |(\boldsymbol{\tau}, \boldsymbol{\epsilon}(\mathbf{v}))| \leq C G_{-1}(\boldsymbol{\tau}, \mathbf{v}; \mathbf{0}) + C G_{-1}(\boldsymbol{\tau}, \mathbf{v}; \mathbf{0})^{\frac{1}{2}} \|\boldsymbol{\tau}\|,$$

where  $G_{-1}(\boldsymbol{\tau}, \mathbf{v}; \mathbf{0})^{\frac{1}{2}}$  denotes the square root of  $G_{-1}(\boldsymbol{\tau}, \mathbf{v}; \mathbf{0})$ . Now, it follows from the Cauchy–Schwarz inequality, (3.19), and (3.13) that

$$\begin{aligned} (\mathcal{A}\boldsymbol{\tau}, \boldsymbol{\tau}) &= (\mathcal{A}\boldsymbol{\tau} - \boldsymbol{\epsilon}(\mathbf{v}), \boldsymbol{\tau}) + (\boldsymbol{\epsilon}(\mathbf{v}), \boldsymbol{\tau}) \\ &\leq \|\mathcal{A}\boldsymbol{\tau} - \boldsymbol{\epsilon}(\mathbf{v})\| \|\boldsymbol{\tau}\| + C G_{-1}(\boldsymbol{\tau}, \mathbf{v}; \mathbf{0}) + C G_{-1}(\boldsymbol{\tau}, \mathbf{v}; \mathbf{0})^{\frac{1}{2}} \|\boldsymbol{\tau}\| \\ &\leq C G_{-1}(\boldsymbol{\tau}, \mathbf{v}; \mathbf{0}) + C G_{-1}(\boldsymbol{\tau}, \mathbf{v}; \mathbf{0})^{\frac{1}{2}} ((\mathcal{A}\boldsymbol{\tau}, \boldsymbol{\tau}) + \|\nabla \cdot \boldsymbol{\tau}\|_{-1, D}^2)^{\frac{1}{2}} \\ &\leq C G_{-1}(\boldsymbol{\tau}, \mathbf{v}; \mathbf{0}) + C G_{-1}(\boldsymbol{\tau}, \mathbf{v}; \mathbf{0})^{\frac{1}{2}} (\mathcal{A}\boldsymbol{\tau}, \boldsymbol{\tau})^{\frac{1}{2}}. \end{aligned}$$

Hence,

$$(\mathcal{A}\boldsymbol{\tau}, \boldsymbol{\tau}) \leq C G_{-1}(\boldsymbol{\tau}, \mathbf{v}; \mathbf{0}),$$

which, together with (3.13), implies the validity of (3.17). With (3.18) and (3.17), it is then easy to see that  $\|\boldsymbol{\epsilon}(\mathbf{v})\|^2$  is also bounded above by the homogeneous functional  $G_{-1}(\boldsymbol{\tau}, \mathbf{v}; \mathbf{0})$ . This completes the proof of the lower bound in (3.14). Since

$$G_{-1}(\boldsymbol{\tau}, \mathbf{v}; \mathbf{0}) \leq G(\boldsymbol{\tau}, \mathbf{v}; \mathbf{0}) \quad \text{and} \quad \|\nabla \cdot \boldsymbol{\tau}\|^2 \leq G(\boldsymbol{\tau}, \mathbf{v}; \mathbf{0}),$$

then the lower bound in (3.15) follows from (3.14). The proof of the theorem is therefore completed.  $\square$

**4. Finite element approximation.** We approximate the minimum of the least-squares functional  $G(\boldsymbol{\sigma}, \mathbf{u}; \mathbf{f})$  in (3.9) using a Rayleigh–Ritz type finite element method. For convenience, we use two-dimensional terminology ( $d = 2$ ). Assuming that the domain  $\Omega$  is a polygon, let  $\mathcal{T}_h$  be a triangulation of  $\Omega$  with triangular elements of size  $\mathcal{O}(h)$  that is regular (see [13]). We restrict ourselves to triangular elements for convenience because extension to either rectangular or a combination of triangular and rectangular elements is straightforward.

Since the homogeneous functional  $G(\boldsymbol{\sigma}, \mathbf{u}; \mathbf{0})$  is equivalent to the  $H(\text{div}; \Omega)$  norm for the stress and the  $H^1$  norm for the displacement by Theorem 3.1 and Korn’s inequality (2.9), it is then natural to approximate the stress (each row) by the standard  $H(\text{div}; \Omega)$  conforming Raviart–Thomas space of order  $k$  (see [20]) and the standard (conforming) continuous piecewise polynomials of degree  $k + 1$  for the displacement:

$$(4.1) \quad \Sigma_h^k = \{\boldsymbol{\tau} \in \mathbf{X}_N : \boldsymbol{\tau}|_K \in RT_k(K)^2 \forall K \in \mathcal{T}_h\} \subset \mathbf{X}_N,$$

$$(4.2) \quad V_h^k = \{\mathbf{v} \in C^0(\Omega)^2 : \mathbf{v}|_K \in P_k(K)^2 \forall K \in \mathcal{T}_h, \mathbf{v} = \mathbf{0} \text{ on } \Gamma_D\} \subset H_D^1(\Omega)^2,$$

where  $RT_k(K)$  is local Raviart–Thomas space of order  $k$  defined by

$$RT_k(K) = P_k(K)^2 + \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} P_k(K),$$

and  $P_k(K)$  is the space of polynomials of degree  $k$  on triangle  $K$ . These spaces have the following approximation properties: if  $k \geq 0$  is an integer and  $l \in (0, k + 1]$ , then

$$(4.3) \quad \inf_{\boldsymbol{\tau} \in \Sigma_h^k} \|\boldsymbol{\sigma} - \boldsymbol{\tau}\|_{H(\text{div}; \Omega)} \leq C h^l (\|\boldsymbol{\sigma}\|_l + \|\nabla \cdot \boldsymbol{\sigma}\|_l)$$

for  $\boldsymbol{\sigma} \in H^l(\Omega)^{2 \times 2} \cap \mathbf{X}_N$  with  $\nabla \cdot \boldsymbol{\sigma} \in H^l(\Omega)^2$ , and

$$(4.4) \quad \inf_{\mathbf{u} \in V_h^{k+1}} \|\mathbf{u} - \mathbf{v}\|_1 \leq C h^l \|\mathbf{u}\|_{l+1}$$

for  $\mathbf{u} \in H^{l+1}(\Omega)^2 \cap H_D^1(\Omega)^2$ .

The finite element approximation for minimizing  $G(\boldsymbol{\sigma}, \mathbf{u}; \mathbf{f})$  in (3.9) on  $\mathcal{V}_B$  becomes: find  $(\boldsymbol{\sigma}_h, \mathbf{u}_h) \in \Sigma_h^k \times V_h^{k+1}$  such that

$$(4.5) \quad G(\boldsymbol{\sigma}_h, \mathbf{u}_h; \mathbf{f}) = \min_{(\boldsymbol{\tau}, \mathbf{v}) \in \Sigma_h^k \times V_h^{k+1}} G(\boldsymbol{\tau}, \mathbf{v}; \mathbf{f}).$$

By Theorem 3.1, (2.9), and the fact that  $\Sigma_h^k \times V_h^{k+1}$  is a subspace of  $\mathcal{V}_B$ , we conclude that (4.5) has a unique solution and is equivalent to the weak form: find  $(\boldsymbol{\sigma}_h, \mathbf{u}_h) \in \Sigma_h^k \times V_h^{k+1}$  such that

$$(4.6) \quad \mathcal{F}(\boldsymbol{\sigma}_h, \mathbf{u}_h; \boldsymbol{\tau}, \mathbf{v}) = (-\mathbf{f}, \nabla \cdot \boldsymbol{\tau}) \quad \forall (\boldsymbol{\tau}, \mathbf{v}) \in \Sigma_h^k \times V_h^{k+1},$$

where the bilinear form  $\mathcal{F}(\cdot; \cdot)$  has the form of

$$\mathcal{F}(\boldsymbol{\sigma}_h, \mathbf{u}_h; \boldsymbol{\tau}, \mathbf{v}) = (\mathcal{A} \boldsymbol{\sigma}_h - \boldsymbol{\epsilon}(\mathbf{u}_h), \mathcal{A} \boldsymbol{\tau} - \boldsymbol{\epsilon}(\mathbf{v})) + (\nabla \cdot \boldsymbol{\sigma}_h, \nabla \cdot \boldsymbol{\tau}).$$

Moreover, the error  $(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \mathbf{u} - \mathbf{u}_h)$  satisfies the orthogonality property

$$(4.7) \quad \mathcal{F}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \mathbf{u} - \mathbf{u}_h; \boldsymbol{\tau}, \mathbf{v}) = 0 \quad \forall (\boldsymbol{\tau}, \mathbf{v}) \in \Sigma_h^k \times V_h^{k+1}.$$

**THEOREM 4.1.** *Assume that the solution,  $(\boldsymbol{\sigma}, \mathbf{u})$ , of (3.9) is in  $H^l(\Omega)^{2 \times 2} \times H^{l+1}(\Omega)^2$  and that the divergence of the stress,  $\nabla \cdot \boldsymbol{\sigma}$ , is in  $H^l(\Omega)^2$ . Let  $k + 1$  be the smallest integer greater than or equal to  $l$ . Then with  $(\boldsymbol{\sigma}_h, \mathbf{u}_h) \in \Sigma_h^k \times V_h^{k+1}$ , the following error estimate holds:*

$$(4.8) \quad \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{H(\text{div}; \Omega)} + \|\mathbf{u} - \mathbf{u}_h\|_1 \leq C h^l (\|\boldsymbol{\sigma}\|_l + \|\nabla \cdot \boldsymbol{\sigma}\|_l + \|\mathbf{u}\|_{l+1}).$$

*Proof.* The proof is a simple consequence of the orthogonality property (4.7) and the approximation properties (4.3) and (4.4) of the finite element spaces  $\Sigma_h^k \times V_h^{k+1}$ .  $\square$

Theorem 3.1 indicates that the bilinear form  $\mathcal{F}(\cdot; \cdot)$  is elliptic and continuous with respect to the  $H(\text{div}; \Omega)$  norm for the stress and the  $H^1$  norm for the displacement. It is then well known that multigrid methods applied to the resulting discrete system (4.6) are optimally convergent (see, e.g., [16, 2, 17, 23]).

It is obvious that the finite element approximation in (4.5) does not preserve the symmetry of the stress. But the finite element approximation of the stress is approximately symmetric. Moreover, one can obtain symmetric stress approximation with the same accuracy as  $\boldsymbol{\sigma}_h$  by simply computing

$$(4.9) \quad \tilde{\boldsymbol{\sigma}}_h = \frac{1}{2} (\boldsymbol{\sigma}_h + \boldsymbol{\sigma}_h^t).$$

It should also be noted that many mixed finite element approaches commonly used produce stress approximations which do not satisfy symmetry exactly (cf. [9, sect. VII.2]).

**COROLLARY 4.2.** *Under the assumptions of Theorem 4.1, we have that*

$$(4.10) \quad \|\boldsymbol{\sigma}_h - \tilde{\boldsymbol{\sigma}}_h\| \leq C h^l (\|\boldsymbol{\sigma}\|_l + \|\nabla \cdot \boldsymbol{\sigma}\|_l + \|\mathbf{u}\|_{l+1})$$

and that

$$(4.11) \quad \|\boldsymbol{\sigma} - \tilde{\boldsymbol{\sigma}}_h\| \leq C h^l (\|\boldsymbol{\sigma}\|_l + \|\nabla \cdot \boldsymbol{\sigma}\|_l + \|\mathbf{u}\|_{l+1}).$$

*Proof.* Since the stress  $\boldsymbol{\sigma}$  is symmetric, by the triangle inequality we have that

$$\|\boldsymbol{\sigma}_h - \boldsymbol{\sigma}_h^t\| = \|(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h) - (\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)^t\| \leq 2\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|$$

and that

$$\|\boldsymbol{\sigma} - \tilde{\boldsymbol{\sigma}}_h\| = \left\| \frac{1}{2}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h) + \frac{1}{2}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)^t \right\| \leq \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|.$$

Now, (4.10) and (4.11) follow from the error bound in (4.8).  $\square$

Nevertheless, there may be a possible reluctance in the engineering community to accept nonsymmetric stress approximation since the symmetry is due to the conservation of angular momentum. In order to directly preserve the symmetry of finite element approximations to the stress tensor, one may enforce the symmetry constraint in the finite element approximation space. To this end, let  $\mathbf{X}_N^s$  denote the symmetric stress space,

$$\mathbf{X}_N^s = \{\boldsymbol{\tau} \in \mathbf{X}_N \mid \boldsymbol{\tau}^t = \boldsymbol{\tau} \text{ in } \Omega\}.$$

A simple and obvious choice is to use continuous piecewise polynomials of degree  $k$  for each component of the symmetric stress:

$$(4.12) \quad \Sigma_h^{k,s} = \{\boldsymbol{\tau} \in C^0(\Omega)^{2 \times 2} \cap \mathbf{X}_N^s : \boldsymbol{\tau}|_K \in P_k(K)^{2 \times 2} \forall K \in \mathcal{T}_h\} \subset \mathbf{X}_N^s.$$

This space has the following approximation property: if  $k \geq 1$  is an integer and  $l \in (0, k]$ , then

$$(4.13) \quad \inf_{\boldsymbol{\tau} \in \Sigma_h^{k,s}} \|\boldsymbol{\sigma} - \boldsymbol{\tau}\|_{H(\text{div}; \Omega)} \leq C h^l \|\boldsymbol{\sigma}\|_{l+1}$$

for  $\boldsymbol{\sigma} \in H^{l+1}(\Omega)^2 \cap \mathbf{X}_N$ . Now, the least-squares finite element approximation problem is to minimize  $G$  over  $\Sigma_h^{k,s} \times V_h^k$ : find  $(\boldsymbol{\sigma}_h, \mathbf{u}_h) \in \Sigma_h^{k,s} \times V_h^k$  such that

$$(4.14) \quad G(\boldsymbol{\sigma}_h, \mathbf{u}_h; \mathbf{f}) = \inf_{(\boldsymbol{\tau}, \mathbf{v}) \in \Sigma_h^{k,s} \times V_h^k} G(\boldsymbol{\tau}, \mathbf{v}; \mathbf{f}).$$

It is easy to see that (4.14) has a unique solution  $(\boldsymbol{\sigma}_h, \mathbf{u}_h)$ , that  $\boldsymbol{\sigma}_h$  is symmetric, and that  $\boldsymbol{\sigma}_h$  has the following error bound:

$$(4.15) \quad \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{H(\text{div}; \Omega)} + \|\mathbf{u} - \mathbf{u}_h\|_1 \leq C h^l (\|\boldsymbol{\sigma}\|_{l+1} + \|\mathbf{u}\|_{l+1})$$

if the solution,  $(\boldsymbol{\sigma}, \mathbf{u})$ , of (3.9) is in  $H^{l+1}(\Omega)^{2 \times 2} \times H^{l+1}(\Omega)^2$ . Note that this estimate is not optimal in the regularity of the displacement. Nevertheless, the nodal elements  $\Sigma_h^{k,s}$  have many fewer local average degrees of freedom than the Raviart–Thomas elements  $\Sigma_h^k$ . Developing a better finite element space of the symmetric stress will be a topic of our further study.

**5. Weakly imposed boundary conditions.** In previous sections, boundary conditions are imposed in the solution space. This leads to least-squares finite element approximations that are much more accurate on the boundary than in the interior of the domain. In the context of the least-squares method, it is natural to treat boundary conditions weakly through boundary functionals. For many applications, this is also convenient. In this section, we study a least-squares functional with boundary terms minimized over a solution space free of boundary conditions. We focus on establishing continuity and ellipticity of this functional here. See [21] for the development of computable finite element approximations and the corresponding iterative solvers based on this functional.

Assume the following nonhomogeneous boundary conditions:

$$(5.1) \quad \mathbf{u} = \mathbf{g} \text{ on } \Gamma_D \quad \text{and} \quad \mathbf{n} \cdot \boldsymbol{\sigma} = \mathbf{h} \text{ on } \Gamma_N.$$

Let

$$\mathcal{V} = \mathbf{X} \times H^1(\Omega)^d,$$

and for  $\mathbf{g} \in H^{1/2}(\Gamma_D)$  and  $\mathbf{h} \in H^{-1/2}(\Gamma_N)$  define the least-squares functional as follows:

$$(5.2) \quad \begin{aligned} \tilde{G}(\boldsymbol{\sigma}, \mathbf{u}; \mathbf{f}, \mathbf{g}, \mathbf{h}) &= \|\mathcal{A}\boldsymbol{\sigma} - \boldsymbol{\epsilon}(\mathbf{u})\|^2 + \|\nabla \cdot \boldsymbol{\sigma} + \mathbf{f}\|^2 \\ &\quad + \|\mathbf{u} - \mathbf{g}\|_{\frac{1}{2}, \Gamma_D}^2 + \|\mathbf{n} \cdot \boldsymbol{\sigma} - \mathbf{h}\|_{-\frac{1}{2}, \Gamma_N}^2 \end{aligned}$$

for  $(\boldsymbol{\sigma}, \mathbf{u}) \in \mathcal{V}$ . The least-squares variational problem is then to minimize  $\tilde{G}$  over  $\mathcal{V}$ : find  $(\boldsymbol{\sigma}, \mathbf{u}) \in \mathcal{V}$  such that

$$(5.3) \quad \tilde{G}(\boldsymbol{\sigma}, \mathbf{u}; \mathbf{f}, \mathbf{g}, \mathbf{h}) = \inf_{(\boldsymbol{\tau}, \mathbf{v}) \in \mathcal{V}} \tilde{G}(\boldsymbol{\tau}, \mathbf{v}; \mathbf{f}, \mathbf{g}, \mathbf{h}).$$

To establish the continuity and ellipticity of the homogeneous least-squares functional  $\tilde{G}(\mathbf{u}, \boldsymbol{\sigma}; \mathbf{0}, \mathbf{0}, \mathbf{0})$  in  $\mathcal{V}$ , we need the trace inequalities (see [15])

$$\begin{aligned} \|u\|_{\frac{1}{2}, \partial\Omega} &\leq \|u\|_1 & \forall u \in H^1(\Omega), \\ \|\mathbf{n} \cdot \mathbf{v}\|_{-\frac{1}{2}, \partial\Omega} &\leq \|\mathbf{v}\|_{H(\text{div}; \Omega)} & \forall \mathbf{v} \in H(\text{div}; \Omega) \end{aligned}$$

and the generalized Korn inequality

$$(5.4) \quad \|\mathbf{v}\|_1 \leq C (\|\boldsymbol{\epsilon}(\mathbf{v})\| + \|\mathbf{v}\|_{0, \Gamma_D}) \quad \forall \mathbf{v} \in H^1(\Omega)^d.$$

**THEOREM 5.1.** *The homogeneous functional  $\tilde{G}(\boldsymbol{\tau}, \mathbf{v}; \mathbf{0}, \mathbf{0}, \mathbf{0})$  is uniformly equivalent to the functional  $M(\mathbf{v}, \boldsymbol{\tau})$ ; i.e., there exists a positive constant  $C$  independent of  $\lambda$  such that*

$$(5.5) \quad \frac{1}{C} M(\boldsymbol{\tau}, \mathbf{v}) \leq \tilde{G}(\boldsymbol{\tau}, \mathbf{v}; \mathbf{0}, \mathbf{0}, \mathbf{0}) \leq C M(\boldsymbol{\tau}, \mathbf{v})$$

holds for all  $(\boldsymbol{\tau}, \mathbf{v}) \in \mathcal{V}$ .

*Proof.* The upper bound in (5.5) follows easily from the triangle inequality, (3.16), and trace inequalities. The proof of the lower bound in (5.5) is the same as that for Theorem 3.1 except the proof on the upper bound of  $|(\boldsymbol{\tau}, \boldsymbol{\epsilon}(\mathbf{v}))|$ . This is because  $\mathbf{v}$

and  $\mathbf{n} \cdot \boldsymbol{\tau}$  do not satisfy any boundary conditions and our new functional has extra boundary terms. Therefore, it suffices to show that

$$(5.6) \quad |(\boldsymbol{\tau}, \boldsymbol{\epsilon}(\mathbf{v}))| \leq C \tilde{G}(\boldsymbol{\tau}, \mathbf{v}; \mathbf{0}, \mathbf{0}, \mathbf{0}) + C \tilde{G}(\boldsymbol{\tau}, \mathbf{v}; \mathbf{0}, \mathbf{0}, \mathbf{0})^{\frac{1}{2}} \|\boldsymbol{\tau}\|.$$

To this end, the triangle, Cauchy–Schwarz, and generalized Korn inequalities give

$$\begin{aligned} \left| \int_{\partial\Omega} \mathbf{v} \cdot (\mathbf{n} \cdot \boldsymbol{\tau}) \, ds \right| &= \left| \int_{\Gamma_D} \mathbf{v} \cdot (\mathbf{n} \cdot \boldsymbol{\tau}) \, ds + \int_{\Gamma_N} \mathbf{v} \cdot (\mathbf{n} \cdot \boldsymbol{\tau}) \, ds \right| \\ &\leq \|\mathbf{v}\|_{\frac{1}{2}, \Gamma_D} \|\mathbf{n} \cdot \boldsymbol{\tau}\|_{-\frac{1}{2}, \Gamma_D} + \|\mathbf{v}\|_{\frac{1}{2}, \Gamma_N} \|\mathbf{n} \cdot \boldsymbol{\tau}\|_{-\frac{1}{2}, \Gamma_N} \\ &\leq \|\mathbf{v}\|_{\frac{1}{2}, \Gamma_D} \|\boldsymbol{\tau}\|_{H(\text{div}; \Omega)} + \|\mathbf{v}\|_1 \|\mathbf{n} \cdot \boldsymbol{\tau}\|_{-\frac{1}{2}, \Gamma_N}. \end{aligned}$$

Now, it follows from the symmetry of  $\boldsymbol{\epsilon}(\mathbf{v})$ ; integration by parts; the triangle, Cauchy–Schwarz, and generalized Korn inequalities; and (3.6) that

$$\begin{aligned} |(\boldsymbol{\tau}, \boldsymbol{\epsilon}(\mathbf{v}))| &= \left| (\boldsymbol{\tau}, \nabla \mathbf{v}) - \left( \frac{\boldsymbol{\tau} - \boldsymbol{\tau}^t}{2}, \nabla \mathbf{v} \right) \right| \\ &= \left| (\nabla \cdot \boldsymbol{\tau}, \mathbf{v}) - \int_{\partial\Omega} \mathbf{v} \cdot (\mathbf{n} \cdot \boldsymbol{\tau}) \, ds + \left( \frac{\boldsymbol{\tau} - \boldsymbol{\tau}^t}{2}, \nabla \mathbf{v} \right) \right| \\ &\leq \left( \|\nabla \cdot \boldsymbol{\tau}\| + \left\| \frac{\boldsymbol{\tau} - \boldsymbol{\tau}^t}{2} \right\| \right) \|\mathbf{v}\|_1 + \|\mathbf{v}\|_{\frac{1}{2}, \Gamma_D} \|\boldsymbol{\tau}\|_{H(\text{div}; \Omega)} + \|\mathbf{v}\|_1 \|\mathbf{n} \cdot \boldsymbol{\tau}\|_{-\frac{1}{2}, \Gamma_N} \\ &\leq C \left( \|\nabla \cdot \boldsymbol{\tau}\| + \|\mathcal{A}\boldsymbol{\tau} - \boldsymbol{\epsilon}(\mathbf{v})\| + \|\mathbf{n} \cdot \boldsymbol{\tau}\|_{-\frac{1}{2}, \Gamma_N} \right) (\|\boldsymbol{\epsilon}(\mathbf{v})\| + \|\boldsymbol{\tau}\|) + \|\mathbf{v}\|_{\frac{1}{2}, \Gamma_D} \|\nabla \cdot \boldsymbol{\tau}\|, \end{aligned}$$

which, together with (3.18), implies (5.6) and, hence, the theorem.  $\square$

**6. Numerical results.** In this section, numerical results for a benchmark problem of linear elasticity taken from [22] are presented. The problem to be considered is given by a quadratic membrane of elastic isotropic material with a circular hole in the center. Traction forces act on the upper and lower edges of the strip. Because of the symmetry of the domain, it suffices to discretize only a fourth of the total geometry. The computational domain is then given by

$$\Omega = \{\mathbf{x} \in \mathbb{R}^2 : 0 < x_1 < 10, 0 < x_2 < 10, x_1^2 + x_2^2 > 1\}$$

(see Figure 6.1). The boundary conditions on the top edge of the computational domain ( $x_2 = 10, 0 < x_1 < 10$ ) are set to  $\boldsymbol{\sigma} \cdot \mathbf{n} = 4.5$ , the boundary conditions on the bottom ( $x_2 = 0, 1 < x_1 < 10$ ) are set to  $(\sigma_{11}, \sigma_{12}) \cdot \mathbf{n} = 0, u_2 = 0$  (symmetry condition), and, finally, the boundary conditions on the left ( $x_1 = 0, 1 < x_2 < 10$ ) are given by  $u_1 = 0, (\sigma_{21}, \sigma_{22}) \cdot \mathbf{n} = 0$  (symmetry condition). The material parameters are  $E = 206900$  for Young’s modulus and  $\nu = 0.29$  for Poisson’s ratio, and their relation with the Lamé constants is given by

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} \quad \text{and} \quad \mu = \frac{E}{2(1+\nu)}.$$

Obviously, the definition of the functional in (3.8) implies

$$G(\boldsymbol{\sigma}_h, \mathbf{u}_h; \mathbf{f}) = \sum_{K \in \mathcal{T}_h} (\|\mathcal{A}\boldsymbol{\sigma}_h - \boldsymbol{\epsilon}(\mathbf{u}_h)\|_{0,K}^2 + \|\nabla \cdot \boldsymbol{\sigma}_h + \mathbf{f}\|_{0,K}^2) =: \sum_{K \in \mathcal{T}_h} G_K(\boldsymbol{\sigma}_h, \mathbf{u}_h; \mathbf{f}).$$

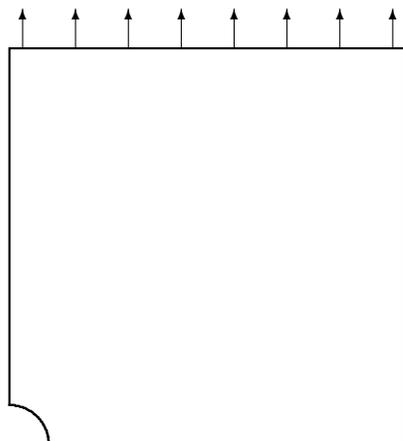


FIG. 6.1. Computational domain and boundary conditions.

TABLE 6.1  
Adaptive finite element approximation ( $k = 1$ ,  $\nu = 0.29$ ).

	# elements	$\dim \Sigma_h^1$	$\dim V_h^2$	Functional	$(\sigma_h)_{22}(1, 0)$
$l = 0$	52	504	224	2.56e-1	9.8830
$l = 1$	115	1130	480	3.78e-2	12.5226
$l = 2$	243	2400	1002	7.61e-2	13.4090
$l = 3$	511	5058	2096	1.62e-3	13.7213
$l = 4$	1069	10600	4366	4.82e-3	13.8259
$l = 5$	2164	21468	8828	1.21e-4	13.8630
$l = 6$	4384	43532	17844	3.51e-5	13.8771
$l = 7$	8678	86190	35302	9.20e-6	13.8912
$l = 8$	17152	170398	69730	2.59e-6	13.8884

Due to the equivalence (3.15), the local contributions  $G_K(\sigma_h, \mathbf{u}_h; \mathbf{f})$  to the least-squares functional constitute an a posteriori error estimator to be used for adaptive refinement (cf. [5]). The results in Table 6.1 are computed on a sequence of adaptively refined meshes based on this error estimator. In each refinement step those triangles with the largest values of  $G_K(\sigma_h, \mathbf{u}_h; \mathbf{f})$  (roughly 25 percent) were refined regularly (by dividing each into four congruent subtriangles). The Raviart–Thomas spaces of order one for the stress approximation are coupled with standard quadratic conforming elements for the displacement ( $\Sigma_h^1 \times V_h^2$  in the terminology of section 4).

Table 6.1 provides a strong indication that the minimum of the functional is inversely proportional to the square of the number of degrees of freedom:

$$\mathcal{F}_h(\sigma_h, \mathbf{u}_h) \sim \frac{1}{(\dim \Sigma_h^1 + \dim V_h^2)^2}.$$

This is the optimal asymptotic convergence rate achievable with piecewise quadratic finite elements. Of particular interest in this example is the stress component  $\sigma_{22}$  at the point  $(1, 0)$ . The size of this stress component is responsible for failure of the

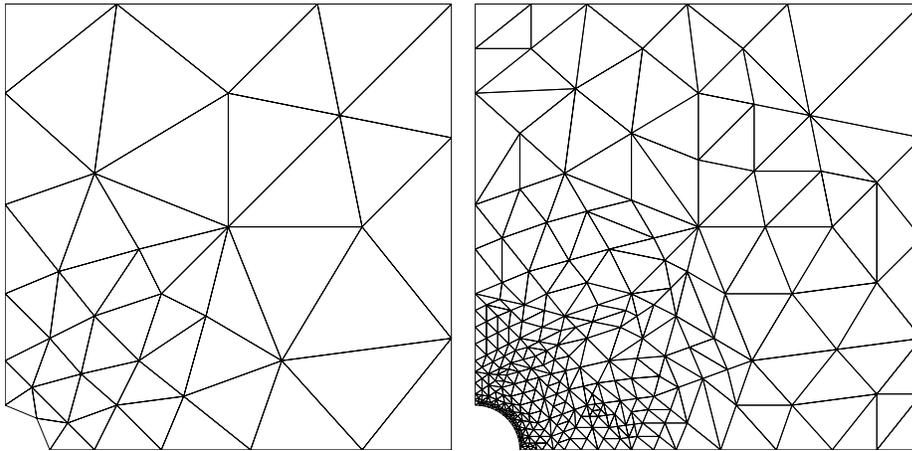


FIG. 6.2. *Initial triangulation and result after three adaptive refinement steps.*

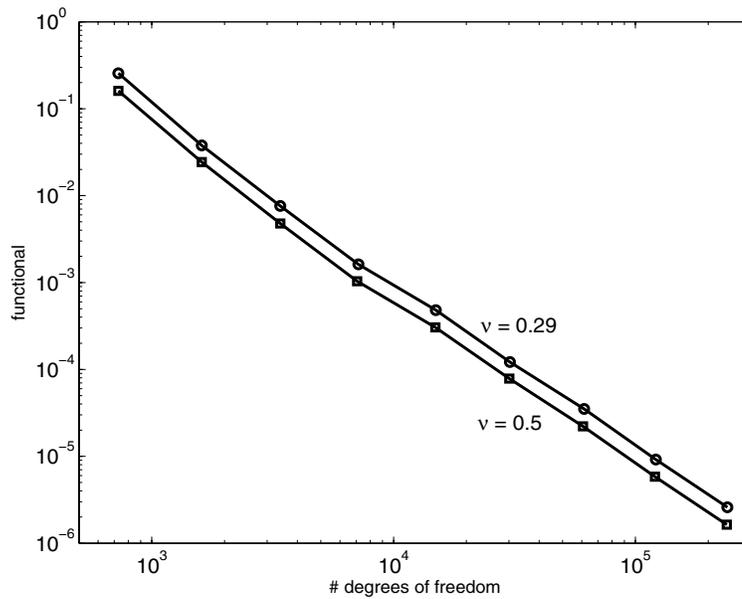


FIG. 6.3. *Adaptive finite element approximation for  $k = 1$  ( $\nu = 0.29, 0.5$ ).*

material at this point. For  $\nu = 0.29$  the value of  $\sigma_{22}(1, 0) = 13.8873$  is given in [22] for a reference solution computed by a polynomial approximation of high degree. The corresponding column in Table 6.1 shows the convergence of the solutions obtained with our least-squares approach to that reference value as the mesh is refined. The initial triangulation and the result of three adaptive refinement steps are shown in Figure 6.2.

The robustness with respect to the incompressible limit can be seen in the doubly logarithmic convergence graphs in Figure 6.3. In addition to the numbers of Table 6.1, the results for the incompressible limit ( $\nu = 0.5$ ) are shown in Figure 6.3.

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