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Asymptotically exact a posteriori error estimators for first-order div least-squares methods in local and global L_2 norm

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ABSTRACT

A new asymptotically exact a posteriori error estimator is developed for first-order div least-squares (LS) finite element methods. Let $(u_h, \boldsymbol{\sigma}_h)$ be LS approximate solution for $(u, \boldsymbol{\sigma} = -A\nabla u)$. Then, $\mathcal{E} = \|A^{-1/2}(\boldsymbol{\sigma}_h + A\nabla u_h)\|_0$ is asymptotically exact a posteriori error estimator for $\|A^{1/2}\nabla(u - u_h)\|_0$ or $\|A^{-1/2}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\|_0$ depending on the order of approximate spaces for $\boldsymbol{\sigma}$ and u. For $\mathcal{E}$ to be asymptotically exact for $\|A^{1/2}\nabla(u - u_h)\|_0$, we require higher order approximation property for $\boldsymbol{\sigma}$, and vice versa. When both $A\nabla u$ and $\boldsymbol{\sigma}$ are approximated in the same order of accuracy, the estimator becomes an equivalent error estimator for both errors. The underlying mesh is only required to be shape regular, i.e., it does not require quasi-uniform mesh nor any special structure for the underlying meshes. Confirming numerical results are provided and the performance of the estimator is explored for other choice of spaces for $(u_h, \boldsymbol{\sigma}_h)$.

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1. Introduction

The purpose of this paper is to introduce new, straightforward a posteriori error estimators for the least-squares (LS) finite element method for second order self-adjoint elliptic partial differential equations proposed in [1,2]. In these papers, the second-order equations are transformed into a system of first-order by introducing a new variable (flux) $\boldsymbol{\sigma} = -A\nabla u$. Least-squares methods based on the first-order system lead to a minimization problem, and the resulting algebraic equations involve a symmetric and positive definite matrix. One of the advantages of LS approaches is that it does not require inf–sup condition [3,4]. As a result, one can choose any conforming finite element spaces as approximate spaces. However, as was explained in [5], optimal rate of convergence for the flux in L_2-norm cannot be obtained without adding the redundant curl equation to the first-order system if the standard continuous piecewise polynomial spaces are used to approximate the dual variable $\boldsymbol{\sigma}$. On the other hand, with $H(\text{div})$ conforming spaces (such as the Raviart–Thomas (RT) spaces[6]) for the dual variable $\boldsymbol{\sigma}$, optimal rate of convergence is achieved for least-squares finite element methods [7]. Bochev and Gunzburger also noted the advantages of using RT spaces over the standard continuous piecewise polynomial spaces when a locally conservative approximation is essential [5,8]. With this as motivation, we will employ such approximation spaces in this paper.

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First-order LS methods approximate the primary variable u and dual variables $\boldsymbol{\sigma} = -A\nabla u$ simultaneously. In general, lowest order approximation spaces are used, i.e. piecewise linear polynomial spaces for u and RT_0 for $\boldsymbol{\sigma}$. However, this leads to approximation of the primary variable with $\mathcal{O}(h^2)$, while the dual variables $\boldsymbol{\sigma}$ are approximated with $\mathcal{O}(h)$. Hence, it is natural to consider different approximation spaces. Indeed, the error estimate in [7] indicates that using the lowest piecewise polynomial space for u and RT_1 for the dual variable approximate both variables with $\mathcal{O}(h^2)$. This motivates us to use different pair of approximations spaces and obtain a posteriori error estimates.

One of the advantages of div LS methods is that the LS functional can be used as an a posteriori error estimator for the natural energy norm. Recently, a modified version of the LS functional, where weight coefficients are introduced to scale the respective residuals, is proposed as a new a posteriori error estimator for these methods in the flux variable [9]. Our estimator uses only one term in the LS functional and the estimator turns out to be asymptotically exact with a certain choices of approximation spaces. Our estimator is of the following form:

$$\mathcal{E}(D) = \|A^{-1/2}(\boldsymbol{\sigma}_h + A\nabla u_h)\|_{0,D},$$

where $(u_h, \boldsymbol{\sigma}_h)$ is the LS solution for $(u, \boldsymbol{\sigma} = -A\nabla u)$ and $D \subseteq \Omega$ is the region of interest. Briefly, when $A\nabla u$ is approximated in higher order approximate spaces, then the estimator is asymptotically exact for $\|A^{-1/2}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\|_{0,D}$ and when $\boldsymbol{\sigma}$ is approximated in higher order, then the estimate is asymptotically exact for $\|A^{1/2}\nabla(u - u_h)\|_{0,D}$. When both $A\nabla u$ and $\boldsymbol{\sigma}$ are approximated in the same order of approximate spaces, then the estimator is equivalent to the error under a mild assumption. Note that one of the advantages of LS methods is that they do not require the inf–sup condition. We use this advantage to choose appropriate approximation spaces for the primary function u and flux variable $\boldsymbol{\sigma}$. We will provide a detailed presentation in Section 4. In our numerical experiments in Section 5, we take $D = \Omega$, and $D = \tau$ where τ is a single element.

Recently, discontinuous Petrov Galerkin (DPG) method is proposed by Demkowicz and Gopalakrishnan [10,11]. Similar to LS approach, the method minimizes a residual of the governing equations in a certain norm. The DPG method has the possibility to locally compute a test space that is close to optimal. It would be interesting topic to modify the a posteriori error estimators developed in this paper for DPF method. We refer the interested readers to [12–18] and references therein for the DPG method and its applications to various problems.

The paper is organized as follows: Section 2 introduces mathematical equations for second-order scalar elliptic partial differential equations; the resulting div least-squares formulation for those equation is then described. In Section 3, we prescribe the finite element spaces and describe the basic properties of the corresponding least-squares approximate solutions. In Section 4, we propose a natural, asymptotically exact a posteriori error estimator for the flux variable $\boldsymbol{\sigma}$ and discuss the properties of the error estimator for different degree pairs of $(u_h, \boldsymbol{\sigma}_h)$. Also, we consider the case when the estimator is reliable and efficient under mild assumption. Finally, in Section 5 we provide numerical results that confirm the preceding analysis and discuss the usefulness of the estimator when asymptotic exactness does not hold.

2. Problem formulation

Let $H^s(\Omega)$ denote the Sobolev space of order s defined on Ω. For $s = 0$, $H^s(\Omega)$ coincides with $L_2(\Omega)$. We shall use the spaces

$$V = H_0^1(\Omega) = \{u \in H^1(\Omega) : u = 0 \text{ on } \partial\Omega\},$$
$$\mathbf{W} = H(\mathrm{div}) = \{\boldsymbol{\sigma} \in (L^2(\Omega))^n : \nabla\cdot\boldsymbol{\sigma} \in L^2(\Omega)\},$$

with norms $\|u\|_1^2 = (u, u) + (\nabla u, \nabla u)$ and $\|\boldsymbol{\sigma}\|_{H(\mathrm{div})}^2 = (\nabla\cdot\boldsymbol{\sigma}, \nabla\cdot\boldsymbol{\sigma}) + (\boldsymbol{\sigma}, \boldsymbol{\sigma})$.

Let Ω be a convex polygonal/polyhedral domain in $\mathbb{R}^n$, $n = 2, 3$, with boundary $\partial\Omega$. Consider

$$\begin{aligned} -\nabla\cdot A\nabla u + cu &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{2.1}$$

where $A = (a_{ij})$ is uniformly symmetric positive definite, and a_{ij}, c and f are smooth functions. We assume the following *a priori* estimate:

$$\|u\|_{2+\delta} \le C\|f\|_\delta, \tag{2.2}$$

for some $\delta > 0$.

By introducing a new variable $\boldsymbol{\sigma} = -A\nabla u \in \mathbf{W}$, we transform the original problem into a system of first-order

$$\begin{aligned} &\boldsymbol{\sigma} + A\nabla u = 0 && \text{in } \Omega, \\ &\nabla\cdot\boldsymbol{\sigma} + c\,u = f && \text{in } \Omega, \\ &u = 0 && \text{on } \partial\Omega. \end{aligned} \tag{2.3}$$

Then, the corresponding least-squares method for the system (2.3) is: Find $u \in V$, $\boldsymbol{\sigma} \in \mathbf{W}$ such that

$$\begin{aligned} b(u, \boldsymbol{\sigma}; v, \mathbf{q}) &\equiv (\nabla\cdot\boldsymbol{\sigma} + c\,u, \nabla\cdot\mathbf{q} + c\,v) + \left(A^{-1}(\boldsymbol{\sigma} + A\nabla u), \mathbf{q} + A\nabla v\right) \\ &= (f, \nabla\cdot\mathbf{q} + c\,v), \end{aligned} \tag{2.4}$$

for all $v \in V$, $\mathbf{q} \in \mathbf{W}$.

3. Finite element approximation

Let $\mathcal{T}_h$ be a regular triangulation of Ω (see [19]) with triangular/tetrahedral elements of size $h = \max\{diam(K); K \in \mathcal{T}_h\}$.

For the approximation space for V, let V_h^r denote the standard continuous piecewise polynomials of degree r. Then, it is well-known that V_h^r has the following approximation property, see [19]: let $k \geq 0$ be an integer and let $l \in [0, r]$

$$\inf_{v \in V_h^r} \|u - v\|_1 \leq C\, h^l\, \|u\|_{l+1}, \tag{3.1}$$

for $u \in H^{l+1}(\Omega) \cap V$. We will use the Scott–Zhang interpolant u_I of u satisfying the above approximation property, see [20].

For the approximation space for $\mathbf{W}$, let $\mathbf{W}_h^k$ denote the standard $H(\mathrm{div})$ conforming Raviart–Thomas space of index k [6]. Then, the Fortin projection $\Pi_h : \mathbf{W} \mapsto \mathbf{W}_h^k$ satisfies

$$\|\boldsymbol{\sigma} - \Pi_h \boldsymbol{\sigma}\|_0 \leq Ch^s \|\boldsymbol{\sigma}\|_s, \tag{3.2}$$

for all $\boldsymbol{\sigma} \in (H^s(\Omega))^n$ for $1 \leq s \leq k+1$, see [21].

Remark 3.1. Throughout this paper, r and k will denote the degree of approximation spaces for the primary variable u and flux $\boldsymbol{\sigma}$ respectively.

The finite element approximation to (2.4) is: Find $u_h \in V_h^r$ and $\boldsymbol{\sigma}_h \in \mathbf{W}_h^k$ such that

$$b(u_h, \boldsymbol{\sigma}_h; v_h, \mathbf{q}_h) = (f, \nabla \cdot \mathbf{q}_h + c\, v_h), \tag{3.3}$$

for all $v_h \in V_h^r$, $\mathbf{q}_h \in \mathbf{W}_h^k$.

The following estimates provide our main technical tool and is provided in [7, Theorem 4.3] and [22, Theorem 5.1].

Theorem 3.1. *Let $(u, \boldsymbol{\sigma})$ and $(u_h, \boldsymbol{\sigma}_h)$ satisfy the equations in (2.4) and (3.3) respectively and $u \in H^2(\Omega)$. Then for sufficiently small h, there exists a constant C independent of $h, u, \boldsymbol{\sigma}, f$ such that*

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_0 \leq C\big(\|\boldsymbol{\sigma} - \Pi_h \boldsymbol{\sigma}\|_0 + h\|\nabla(u - u_I)\|_1\big), \tag{3.4}$$

and

$$\|\nabla(u - u_h)\|_0 \leq C\big(\|\nabla(u - u_I)\|_0 + h\|\boldsymbol{\sigma} - \Pi_h \boldsymbol{\sigma}\|_{H(\mathrm{div})}\big). \tag{3.5}$$

4. Asymptotically exact a posteriori error estimator: residual type

We develop an a posteriori error estimator which is asymptotically exact for the error for the flux ($\boldsymbol{\sigma} = -A\nabla u$). If an estimator $\mathcal{E}$ converges to the true error in the limit $h \to 0$, then the estimator is said to be asymptotically exact. Also, an estimator $\mathcal{E}$ is said to be equivalent to the error $|\,\|e\|\,|$ if there exists reliability and efficiency constants C_{ref} and C_{eff} such that

$$C_{\mathrm{eff}} \cdot \mathcal{E} \leq |\,\|e\|\,| \leq C_{ref} \cdot \mathcal{E}.$$

In order to do construct asymptotically exact error estimators, we will choose different approximate spaces for u and $\boldsymbol{\sigma}$. For the remainder of this paper, we set

$$\mathcal{E}(D) = \|A^{-1/2}(\boldsymbol{\sigma}_h + A\nabla u_h)\|_{0,D},$$

where $D \subseteq \Omega$. For the construction of asymptotically exact a posteriori error estimators in Sections 4.1 and 4.2, we assume $u \in H^{2+\delta}(\Omega)$ for some $\delta > 0$ satisfying the a priori estimate (2.2).

4.1. Asymptotically exact estimators for $\|A^{1/2}\nabla(u - u_h)\|_0$

In order to construct an asymptotically exact estimator for $\|A^{1/2}\nabla(u - u_h)\|_0$, we take $k = r$. Note that with this choice of approximate spaces, $\boldsymbol{\sigma}_h$ provides a higher order approximation for $\boldsymbol{\sigma}$ $(= -A\nabla u)$ compared to $A\nabla u_h$. We present our result for $k = r = 1$. Other cases are a straightforward extension of this result.

Let $0 < \epsilon \ll \delta$ be a fixed constant. Using the continuous piecewise linear functions (i.e. $r = 1$) to approximate u, it is well-known that $\|\nabla(u - u_h)\|_0 \sim \mathcal{O}(h)$. Thus, we assume for sufficiently small h,

$$h^{1+\epsilon}|u|_2 \leq \|\nabla(u - u_h)\|_0. \tag{4.1}$$

The above bound is proved in [23]. Also, we assume that the mesh size h is small enough to have

$$h^{\delta - 2\epsilon}|u|_{2+\delta} \leq |u|_2. \tag{4.2}$$

Using the approximation property (3.2) with $k = 1$, and by combining (4.1) and (4.2), we have

$$\begin{aligned}\|\boldsymbol{\sigma} - \Pi_h\boldsymbol{\sigma}\|_0 &\le Ch^{1+\delta}|\boldsymbol{\sigma}|_{1+\delta} \le Ch^{1+\delta}|u|_{2+\delta} \le Ch^{1+2\epsilon}|u|_2 \\ &\le Ch^{\epsilon}\|\nabla(u-u_h)\|_0. \end{aligned} \tag{4.3}$$

Using uniform boundedness of A, (3.4), and (4.3), we have

$$\begin{aligned}\|A^{-1/2}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\|_0 &\le C\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_0 \\ &\le C\|\boldsymbol{\sigma} - \Pi_h\boldsymbol{\sigma}\|_0 + Ch\|\nabla(u-u_h)\|_0 \\ &\le C(h^{\epsilon}+h)\|\nabla(u-u_h)\|_0 \\ &\le Ch^{\epsilon}\|A^{1/2}\nabla(u-u_h)\|_0 \\ &= m_\Omega(h)\|A^{1/2}\nabla(u-u_h)\|_0, \end{aligned} \tag{4.4}$$

where $m_\Omega(h) = Ch^{\epsilon}$. In short, we have

$$\|A^{-1/2}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\|_0 \le m_\Omega(h)\|A^{1/2}\nabla(u-u_h)\|_0, \tag{4.5}$$

where $m_\Omega(h) = Ch^{\epsilon}$. This plays a key role to construct asymptotically exact a posteriori error estimators in the following theorem.

Theorem 4.1. *Let $k = r = 1$ and fix $0 < \epsilon \ll \delta$. There exist a constant $C = C(k, n, a_{ij}, c, \epsilon)$ such that, for h small enough,*

$$\frac{1}{1+m_\Omega(h)}\mathcal{E}(\Omega) \le \|A^{1/2}\nabla(u-u_h)\|_{L_2(\Omega)} \tag{4.6}$$

and if $m_\Omega(h) < 1$,

$$\frac{1}{1+m_\Omega(h)}\mathcal{E}(\Omega) \le \|A^{1/2}\nabla(u-u_h)\|_{L_2(\Omega)} \le \frac{1}{1-m_\Omega(h)}\mathcal{E}(\Omega), \tag{4.7}$$

where

$$m_\Omega(h) = Ch^{\epsilon}.$$

The estimator is asymptotically exact as $h \to 0$ since $m_\Omega(h) \to 0$ as $h \to 0$.

Proof. By the triangle inequality with $\boldsymbol{\sigma} + A\nabla u = 0$, and (4.5), we have

$$\begin{aligned}\|A^{-1/2}(\boldsymbol{\sigma}_h + A\nabla u_h)\|_0 &\le \|A^{-1/2}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\|_0 + \|A^{1/2}\nabla(u-u_h)\|_0 \\ &\le (m_\Omega(h)+1)\|A^{1/2}\nabla(u-u_h)\|_0. \end{aligned}$$

Hence, we have

$$\frac{1}{1+m_\Omega(h)}\|A^{-1/2}(\boldsymbol{\sigma}_h + A\nabla u_h)\|_0 \le \|A^{1/2}\nabla(u-u_h)\|_0. \tag{4.8}$$

This proves the lower bound.

For the upper bound, we assume $m_\Omega(h) = Ch^{\epsilon} < 1$. This is clearly true when h is small. By the triangle inequality, using $\boldsymbol{\sigma} + A\nabla u = 0$ and (4.5), we have

$$\begin{aligned}\|A^{1/2}\nabla(u-u_h)\|_0 &\le \|A^{-1/2}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h + A\nabla(u-u_h))\|_0 + \|A^{-1/2}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\|_0 \\ &\le \|A^{-1/2}(\boldsymbol{\sigma}_h + A\nabla u_h)\|_0 + m_\Omega(h)\|A^{1/2}\nabla(u-u_h)\|_0. \end{aligned} \tag{4.9}$$

Thus, we have

$$\|A^{1/2}\nabla(u-u_h)\|_0 \le \frac{1}{1-m_\Omega(h)}\|A^{-1/2}(\boldsymbol{\sigma}_h + A\nabla u_h)\|_0. \tag{4.10}$$

This completes the proof. □

Remark 4.1. Note that our theorem has many similarities with the one developed in [24]. The key ingredient is a construction of a better approximation for $A\nabla u$ than $A\nabla u_h$. In the LS formulation, the direct approximation $\boldsymbol{\sigma}_h$ for $\boldsymbol{\sigma} = -A\nabla u$ with $k = r$ provides the higher order approximate solution and this is the key ingredient for our approach. One important feature of our estimator is that it allows highly graded meshes and still produces asymptotically exactness under the condition that the mesh size $h \to 0$.

An interesting fact about the estimator $\|A^{-1/2}(\boldsymbol{\sigma}_h + A\nabla u_h)\|_0$ is that it can also be used as a local estimator for the error for a region of interest $D \subset \Omega$. Let D be a fixed region of interest. In order to provide local estimator, we need to introduce two constant R_1, R_2 defined as follows:

$$R_1(D) = \frac{\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,\Omega}}{\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,D}} \geq 1, \tag{4.11}$$

and

$$R_2(D) = \frac{\|A^{1/2}\nabla(u - u_h)\|_{0,\Omega}}{\|A^{1/2}\nabla(u - u_h)\|_{0,D}} \geq 1. \tag{4.12}$$

Note that for any fixed $\epsilon > 0$ and for sufficiently small h, we have

$$h^{\frac{\epsilon}{2}}\|A^{1/2}\nabla(u - u_h)\|_{0,\Omega} \leq \|A^{1/2}\nabla(u - u_h)\|_{0,D}. \tag{4.13}$$

Using uniform boundedness of A, (3.4), and (4.5), we have

$$\begin{aligned}\|A^{-1/2}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\|_{0,D} &\leq C\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,D} = \frac{C}{R_1(D)}\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,\Omega} \\ &\leq \frac{C}{R_1(D)}\|\boldsymbol{\sigma} - \Pi_h\boldsymbol{\sigma}\|_{0,\Omega} + \frac{C}{R_1(D)}h\|\nabla(u - u_h)\|_0 \\ &\leq \frac{C}{R_1(D)}(h^\epsilon + h)\|\nabla(u - u_h)\|_0 \\ &\leq \frac{C}{R_1(D)}h^\epsilon\|A^{1/2}\nabla(u - u_h)\|_0 = C\frac{R_2(D)}{R_1(D)}h^\epsilon\|A^{1/2}\nabla(u - u_h)\|_{0,D} \\ &= m_D(h)\|A^{1/2}\nabla(u - u_h)\|_{0,D},\end{aligned} \tag{4.14}$$

where $m_D(h) = C\frac{R_2(D)}{R_1(D)}h^\epsilon$. Note that using (4.13), we have

$$m_D(h) \to 0 \quad \text{as } h \to 0.$$

In short, we have

$$\|A^{-1/2}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\|_{0,D} \leq m_D(h)\|A^{1/2}\nabla(u - u_h)\|_{0,D}, \tag{4.15}$$

where $m_D(h) \to 0$ as $h \to 0$.

Now, we present a local asymptotically-exact error estimator.

Theorem 4.2. *Let $k = r = 1$ and fix $0 < \epsilon \ll \delta$ and $D \subset \Omega$. There exist a constant $C = C(k, n, a_{ij}, c, \epsilon, D)$ such that, for h small enough,*

$$\frac{1}{1 + m_D(h)}\mathcal{E}(D) \leq \|A^{1/2}\nabla(u - u_h)\|_{0,D} \tag{4.16}$$

and if $m_D(h) < 1$,

$$\frac{1}{1 + m_D(h)}\mathcal{E}(D) \leq \|A^{1/2}\nabla(u - u_h)\|_{L_2(D)} \leq \frac{1}{1 - m_D(h)}\mathcal{E}(D), \tag{4.17}$$

where

$$m_D(h) = C\frac{R_2(D)}{R_1(D)}h^\epsilon.$$

The estimator is asymptotically exact as $h \to 0$ since $m_D(h) \to 0$ as $h \to 0$.

Proof. By the triangle inequality, using $\boldsymbol{\sigma} + A\nabla u = 0$ and (4.15), we have

$$\begin{aligned}\|A^{1/2}\nabla(u - u_h)\|_{0,D} &\leq \|A^{-1/2}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h + A\nabla(u - u_h))\|_{0,D} + \|A^{-1/2}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\|_{0,D} \\ &\leq \|A^{-1/2}(\boldsymbol{\sigma}_h + A\nabla u_h)\|_{0,D} + m_D(h)\|A^{1/2}\nabla(u - u_h)\|_{0,D}.\end{aligned} \tag{4.18}$$

Thus, we have

$$\|A^{1/2}\nabla(u - u_h)\|_{0,D} \leq \frac{1}{1 - m_D(h)}\|A^{-1/2}(\boldsymbol{\sigma}_h + A\nabla u_h)\|_{0,D}. \tag{4.19}$$

This proves the upper bound. The lower bound can be obtained in a similar manner. This completes the proof. □

4.2. Asymptotically exact estimators for $\|A^{-1/2}(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)\|_0$

In order to construct asymptotically exact estimators for $\|A^{-1/2}(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)\|_0$, we take $r > k+1$. Note that with the choice of approximate spaces, $A\nabla u_h$ provides the higher order approximation for $\boldsymbol{\sigma}$ $(= -A\nabla u)$ compared to $\boldsymbol{\sigma}_h$. We present our result for $r = 2$ and $k = 0$. It is straightforward extension for other cases. Let $0 < \epsilon \ll \delta$ be a fixed constant, where δ is defined in (2.2).

Using the lowest RT elements (i.e. $k = 0$) to approximate $\boldsymbol{\sigma}$, it is well-known that $\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_0 \sim \mathcal{O}(h)$. Thus, we assume that for sufficiently small h,

$$h^{1+\epsilon}|\boldsymbol{\sigma}|_1 \le \|A^{-1/2}(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)\|_0. \tag{4.20}$$

Also, we assume that the mesh size h is small enough to have

$$h^{\delta-2\epsilon}(|\boldsymbol{\sigma}|_{1+\delta} + \|f\|_\delta) \le |\boldsymbol{\sigma}|_1. \tag{4.21}$$

Using (3.5), approximation properties, (2.3), and combining (4.20) and (4.21), we have

$$\begin{aligned}\|A^{1/2}\nabla(u-u_h)\|_0 &\le C\|\nabla(u-u_I)\|_0 + Ch\|\boldsymbol{\sigma}-\Pi_h\boldsymbol{\sigma}\|_{H(\mathrm{div})}\\ &\le Ch^{1+\delta}(|u|_{2+\delta} + |\boldsymbol{\sigma}|_1 + |\nabla\cdot\boldsymbol{\sigma}|_\delta) \le Ch^{1+\delta}(|\boldsymbol{\sigma}|_{1+\delta} + \|f\|_\delta)\\ &\le Ch^{1+2\epsilon}|\boldsymbol{\sigma}|_1 \le Ch^{\epsilon}\|A^{-1/2}(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)\|_0.\end{aligned} \tag{4.22}$$

We these assumptions, we present an asymptotically exact a posteriori error estimator for $\|A^{-1/2}(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)\|_0$.

Theorem 4.3. *Let $k = 0$ and $r = 2$ and fix $0 < \epsilon < 1$. There exist a constant $C = C(k, n, a_{ij}, c, \epsilon)$ such that, for h small enough,*

$$\frac{1}{1+m_\Omega(h)}\mathcal{E}(\Omega) \le \|A^{-1/2}(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)\|_0 \tag{4.23}$$

and if $m_\Omega(h) < 1$,

$$\frac{1}{1+m_\Omega(h)}\mathcal{E}(\Omega) \le \|A^{-1/2}(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)\|_0 \le \frac{1}{1-m_\Omega(h)}\mathcal{E}(\Omega), \tag{4.24}$$

where

$$m_\Omega(h) = Ch^\epsilon.$$

The estimator is asymptotically exact as $h \to 0$ since $m_\Omega(h) \to 0$ as $h \to 0$.

Proof. Using the triangle inequality and (4.22) with $m_\Omega(h) = Ch^\epsilon$, we have

$$\begin{aligned}\|A^{-1/2}(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)\|_0 &\le \|A^{-1/2}(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h + A\nabla(u-u_h))\|_0 + \|A^{1/2}\nabla(u-u_h)\|_0\\ &\le \|A^{-1/2}(\boldsymbol{\sigma}_h + A\nabla u_h)\|_0 + m_\Omega(h)\|A^{-1/2}(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)\|_0.\end{aligned}$$

Thus, we have

$$\|A^{-1/2}(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)\|_0 \le \frac{1}{1-m_\Omega(h)}\mathcal{E}(\Omega).$$

For the lower bound, by the triangle inequality and using $\boldsymbol{\sigma} = -A\nabla u$, we have

$$\begin{aligned}\|A^{-1/2}(\boldsymbol{\sigma}_h + A\nabla u_h)\|_0 &\le \|A^{-1/2}(\boldsymbol{\sigma}_h - \boldsymbol{\sigma})\|_0 + \|A^{-1/2}(\boldsymbol{\sigma} + A\nabla u_h)\|_0\\ &= \|A^{-1/2}(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)\|_0 + \|A^{1/2}\nabla(u-u_h)\|_0\\ &\le \|A^{-1/2}(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)\|_0 + m_\Omega(h)\|A^{-1/2}(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)\|_0\\ &= (1+m_\Omega(h))\|A^{-1/2}(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)\|_0.\end{aligned}$$

Hence, we have

$$\frac{1}{1+m_\Omega(h)}\|A^{-1/2}(\boldsymbol{\sigma}_h + A\nabla u_h)\|_0 \le \|A^{-1/2}(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)\|_0.$$

This completes the proof. □

For the local error estimators, note that for any fixed $\epsilon > 0$, we have for sufficiently small h,

$$h^{\epsilon/2}\|A^{-1/2}(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)\|_{0,\Omega} \le \|A^{-1/2}(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)\|_{0,D}. \tag{4.25}$$

With the above inequality and following the similar procedure in (4.14) we obtain

$$\|A^{1/2}\nabla(u-u_h)\|_{0,D} \le m_D(h)\|A^{-1/2}(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)\|_{0,D}, \tag{4.26}$$

where $m_D(h) = C\frac{R_1(D)}{R_2(D)}h^\epsilon \to 0$ as $h \to 0$. The following can be obtained following Theorem 4.2.

Theorem 4.4. *Let $k = 0$ and $r = 2$ and fix $0 < \epsilon \ll \delta$ and $D \subset \Omega$. There exist a constant $C = C(k, n, a_{ij}, c, \epsilon, D)$ such that, for h small enough,*

$$\frac{1}{1 + m_D(h)} \mathcal{E}(D) \le \|A^{-1/2}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\|_{L_2(D)} \tag{4.27}$$

and if $m_D(h) < 1$,

$$\frac{1}{1 + m_D(h)} \mathcal{E}(D) \le \|A^{-1/2}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\|_{0,D} \le \frac{1}{1 - m_D(h)} \mathcal{E}(D), \tag{4.28}$$

where

$$m_D(h) = C \frac{R_1(D)}{R_2(D)} h^{\epsilon}.$$

The estimator is asymptotically exact as $h \to 0$ since $m_D(h) \to 0$ as $h \to 0$.

4.3. Equivalent a posteriori error estimator

We now consider the case $k + 1 = r$, i.e. both $A\nabla u$ and $\boldsymbol{\sigma}$ are approximated in the same order of accuracy. When using a common pairing of div LS spaces $(P_{k+1} - RT_k)$, our error estimator loses asymptotic exactness for either error. This is not unreasonable, as our analysis requires one of the pairings to be "higher-order" (even if it is a mere power of epsilon). The more interesting question is whether or not $\mathcal{E}$ produces a reliable and efficient estimator for either error. We provide an argument that $\|A^{-1/2}(\boldsymbol{\sigma}_h + A\nabla u_h)\|$ is an equivalent error estimators for both $\|A^{1/2}\nabla(u - u_h)\|$ and $\|A^{-1/2}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\|$ under a mild assumption. Our assumption is of the following form:

$$\frac{1}{m}\|A^{1/2}\nabla(u - u_h)\|_{0,D} \le \|A^{-1/2}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\|_{0,D} \le m\|A^{1/2}\nabla(u - u_h)\|_{0,D}, \tag{4.29}$$

where m, $0 < m < 1$ is a fixed constant independent of h. Note that both $\boldsymbol{\sigma}_h$ and $A\nabla u_h$ are approximate solutions for the flux($\boldsymbol{\sigma} = -A\nabla u$), modulo sign. Thus, it is natural to expect that the direct approximation $\boldsymbol{\sigma}_h$ is more accurate approximation for $\boldsymbol{\sigma}$ compared to $A\nabla u_h$ when the approximate spaces for the both variable are of the same order, i.e. $k + 1 = r$ and this is reflected in our assumption (4.29). Our numerical experiments confirm this. The above inequality is the same inequality as in (4.15) and (4.26) except the fact $0 < m < 1$ is a constant, not $m \to 0$ as $h \to 0$. We want to remind the reader that $m \to 0$ as $h \to 0$ in (4.15) and (4.26) since one variable is approximated higher order than the other variable.

Now, using the second inequality in our assumption (4.29) and using the same procedure in the proof of Theorem 4.2, we have

Theorem 4.5. *Let $k + 1 = r$ and assume (4.29). Then,*

$$\frac{1}{1 + m} \mathcal{E}(D) \le \|A^{1/2}\nabla(u - u_h)\|_{0,D} \le \frac{1}{1 - m} \mathcal{E}(D). \tag{4.30}$$

Using the first inequality in our assumption (4.29) and following the same procedure in the proof of Theorem 4.4.

Theorem 4.6. *Under the same assumption as Theorem 4.5, we have*

$$\frac{1}{1 + m} \mathcal{E}(D) \le \|A^{-1/2}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\|_{0,D} \le \frac{1}{1 - m} \mathcal{E}(D). \tag{4.31}$$

Remark 4.2. We observe that our estimator is similar in philosophy to the estimator developed for conforming linear finite elements by Cai and Zhang in [25]. In that paper, a weighted least-squares projection into $H(\text{div})$ is used to recover the flux which is compared with the numerical flux as an estimator. In the div LS method the flux is "reconstructed" directly as part of the solution process. Additionally, in the spirit of [26], we believe that reliability, not asymptotic exactness, is a realistic goal for "function recovery" when trying to construct an estimator for $\|A^{-1/2}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\|_0$.

5. Numerical experiments

In this section, we provide numerical experiments confirming the asymptotic exactness of our estimator with appropriate approximation degrees k and r, and investigate the performance of the estimator in two dimensional spaces. For higher dimensional cases, the degree of freedom (DOF) has to grow exponentially. We refer to [27,28], where the difficulty is overcome by using canonical tensor decomposition combined with Chebyshev spectral differentiation.

We define the "effectivity index" of an estimator in a standard way, as the ratio of our estimator $\mathcal{E}(\Omega)$ to the ratio of the true error. In our experiments, we approximate $-\Delta u = f$, with homogeneous Dirichlet boundary conditions. In the first

Table 5.1
Effectivity indices for a uniform mesh and smooth solution, $P_1 - RT_1$.

# of elements	Eff. index, $\nabla(u - u_h)$	Eff. index, $\boldsymbol{\sigma} - \boldsymbol{\sigma}_h$.
8	1.089124	0.133016
32	1.017700	0.073681
128	1.004440	0.035210
512	1.001111	0.017401
2048	1.000278	0.008675
8192	1.000069	0.004334

Table 5.2
Effectivity indices for a uniform mesh and smooth solution, $P_2 - RT_0$.

# of elements	Eff. index, $\nabla(u - u_h)$	Eff. index, $\boldsymbol{\sigma} - \boldsymbol{\sigma}_h$.
8	0.425977	1.064655
32	0.419813	1.056767
128	0.205775	1.014065
512	0.102347	1.003508
2048	0.051105	1.000876
8192	0.025544	1.000219

Table 5.3
Effectivity indices for uniform mesh and smooth solution, $P_1 - RT_0$.

# of elements	Eff. index, $\nabla(u - u_h)$	Eff. index, $\boldsymbol{\sigma} - \boldsymbol{\sigma}_h$.
8	1.055975	0.568269
32	0.910413	0.565012
128	0.867158	0.532794
512	0.856389	0.524865
2048	0.853699	0.522891
8192	0.853027	0.522397

two examples, the domain $\Omega = [0, 1] \times [0, 1]$. In the final example, the domain Ω is $[-1, 1] \times [-1, 1]$, again with zero Dirichlet boundary conditions.

Example 1 (*Smooth Solution,* $u = \sin \pi x \sin \pi y$)**.** We first present a study of the estimator on uniform meshes. Here is a table of effectivities for uniform meshes of size 2^{-i}, $i = 1, 2, \ldots, 6$, for $-\Delta u = f$, u_h approximated using the $P_1 - RT_1$ pair. As predicted by Theorem 4.1, our estimator is asymptotically exact for $\|\nabla(u - u_h)\|$. When the estimator is asymptotically exact for $\|\nabla(u-u_h)\|_0$, it is of order $\mathcal{O}(h)$. On the other hand, $\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_0$ is of order $\mathcal{O}(h^2)$. Hence, $\mathcal{E}$ would be a poor estimator for $\boldsymbol{\sigma} - \boldsymbol{\sigma}_h$ and our numerical result confirms this.

We follow with another uniform study, on the same model problem, employing the $P_2 - RT_0$ pair. As predicted in Theorem 4.3, the estimator is asymptotically exact for $\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_0$. When the estimator is asymptotically exact for $\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_0$, it is of order $\mathcal{O}(h)$. Hence, it would be a poor estimator for $\|\nabla(u - u_h)\|_0$ since it is of order $\mathcal{O}(h^2)$.

Here we present the results for our estimator for $P_1 - RT_0$. Clearly, the estimators is not asymptotically exact for either error component, but the results suggest equivalence and reliability for either error.

Remark 5.1. Our proposed estimator behaves identically on this example for uniformly refined quadrilateral meshes with the corresponding choice of $Q_r - BDM_k$ spaces. We obtain asymptotic exactness in the corresponding variable when $k+1 > r$ (or vice-versa), and equivalence (for either error) when $k + 1 = r$, see Tables 5.1–5.3.

Example 2 (*Smooth Solution, Adaptive Algorithm,* $u = (1-x)^4 x \sin(2\pi y^4)$)**.** This example uses $\mathcal{E}(\tau)$ to drive a basic adaptive algorithm using the Dörfler mark and refine strategy [29] for the adaptive algorithm and Delaunay edge-swaps after mesh refinement. The solution, $u = (1 - x)^4 x \sin(2\pi y^4)$, is smooth, but has significant local features. The initial mesh is uniform with $h = 0.5$. The algorithm terminates when less than 1% (normalized) global error is estimated. Fig. 5.1 gives effectivity indices for our estimator, while Fig. 5.2 shows the final adapted mesh.

In Table 5.4 we present local element effectivity statistics, $\mathcal{E}(\tau)/\|\nabla(u-u_h)\|_{L_2(\tau)}$, at the final adaptive level. As illustrated in Fig. 5.2, some mesh elements are quite coarse, but the results are excellent.

We repeated the adaptive experiment, this time employing $P_1 - RT_0$, but again using $\mathcal{E}(\tau)$ to drive an adaptive experiment. The statistics for $\nabla(u - u_h)$ on the final refined mesh are given in Table 5.5.

Example 3 (*Rough Adaptive Solution,* $u = (x^2 - 1)(y^2 - 1)(x^2 + y^2)^{0.51}$)**.** In this example, Ω is $[-1, 1] \times [-1, 1]$ with zero Dirichlet boundary conditions. The solution $u = (x^2 - 1)(y^2 - 1)(x^2 + y^2)^{0.51}$ is not smooth near the origin. We again use the same adaptive algorithm as in Example 2, driven by $\mathcal{E}(\tau)$.

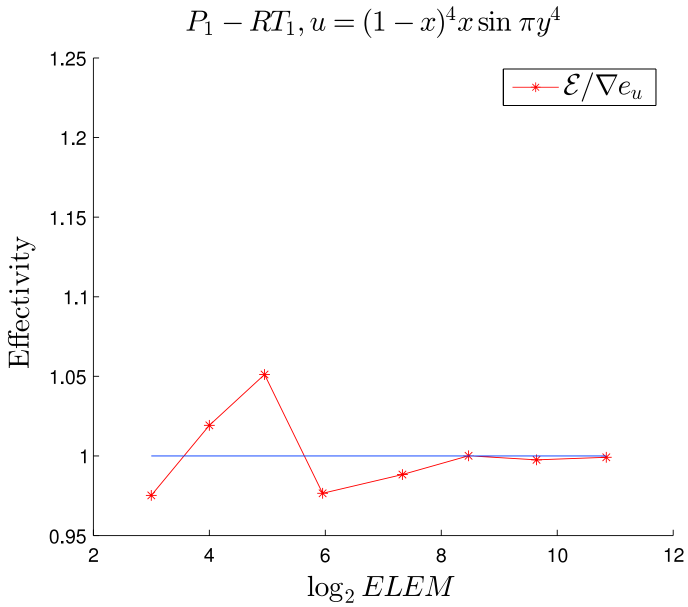

Fig. 5.1. Effectivity indices, adapted mesh, smooth solution, $P_1 - RT_1$.

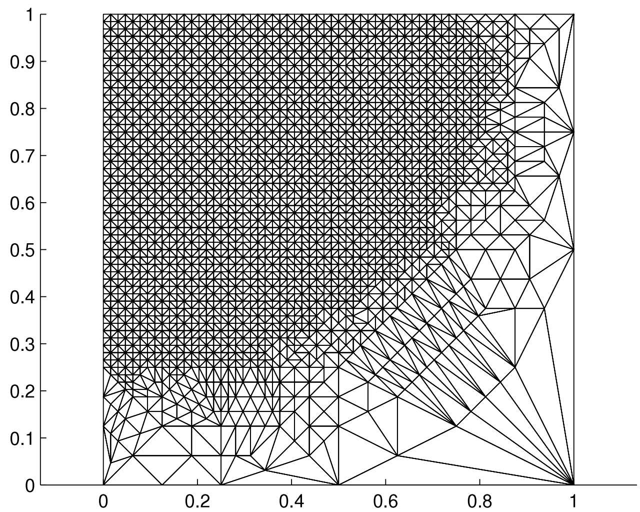

Fig. 5.2. Adapted mesh, smooth solution, $P_1 - RT_1$.

Table 5.4
Effectivity statistics, adapted mesh, smooth solution, $P_1 - RT_1$.

Max. eff	Min. eff	Mean eff	Median eff	Std. deviation
2.427740	0.901903	1.001792	1.000076	0.032406

Table 5.5
Effectivity statistics, adapted mesh, smooth solution, $P_1 - RT_0$.

Max. eff	Min. eff	Mean eff	Median eff	Std. deviation
6.011135	0.368994	1.394727	1.388164	0.216872

In Fig. 5.3, we show the effectivity index for our estimator at each level. The convergence rate(not shown) versus the number of degrees of freedom is consistent with an accurately guided adaptive algorithm.

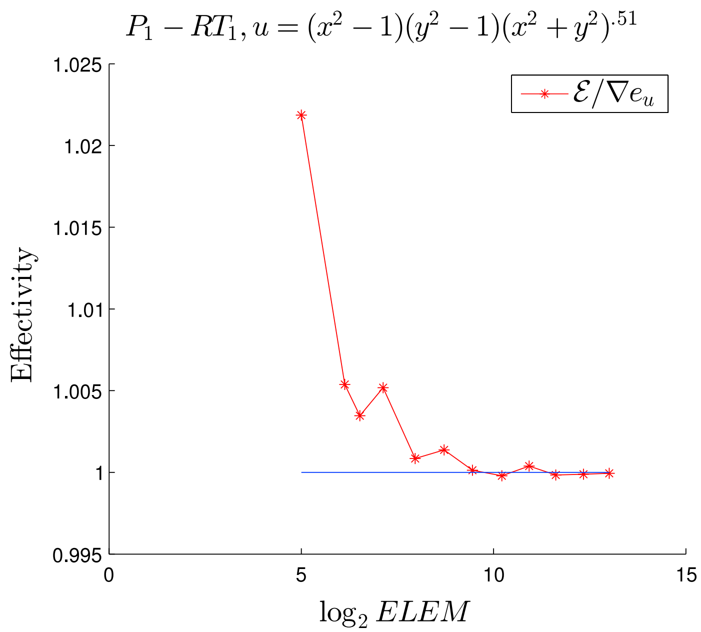

Fig. 5.3. Effectivity indices, adaptive rough solution, $P_1 - RT_1$.

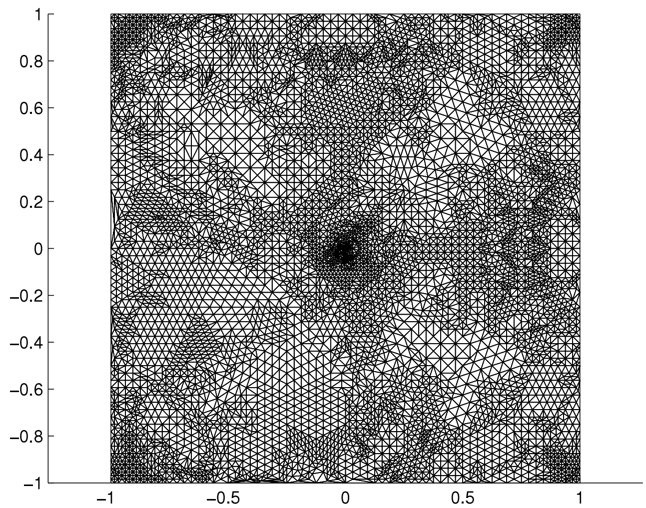

Fig. 5.4. Final adapted mesh, $P_1 - RT_1$, "Rough" solution.

Table 5.6
Effectivity statistics, adapted mesh, rough solution, $P_1 - RT_1$.

Max. eff	Min. eff	Mean eff	Median eff	Std. deviation
1.141773	0.856132	1.000002	0.999956	0.008080

In Fig. 5.4, we show the final adapted mesh, with the expected refinement near the singularity at the origin.

The local (on an element τ) effectivity statistics, given in Table 5.6, demonstrate the local asymptotic properties and robustness of the estimator.

In Table 5.7, we give the effectivity statistics for our "rough" model problem, using $P_2 - RT_0$, at the final mesh level.

When the $P_1 - RT_0$ spaces are used for this example problem, as predicted in Section 4.3, the estimator provides a reliable and efficient estimator, as shown in Fig. 5.5.

As in the earlier examples, the estimator is not asymptotically exact, but clearly still useful. This can be observed from the local statistics are given in Table 5.8, especially in the fairly small standard deviation of the estimator.

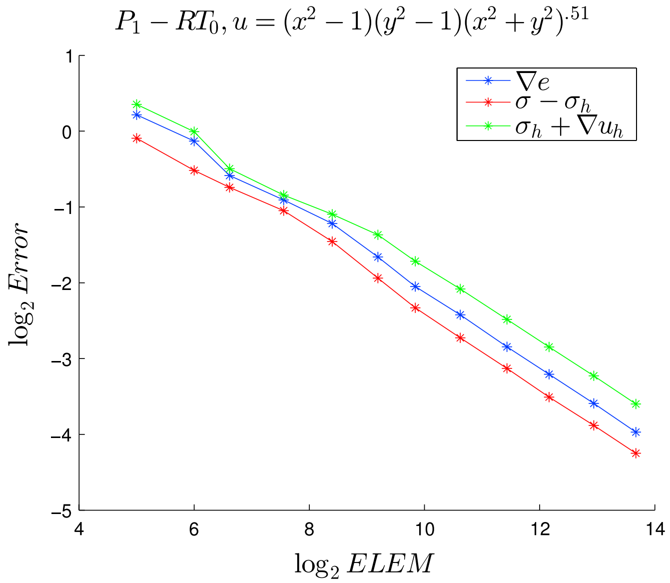

Fig. 5.5. Real and estimated errors, rough adaptive solution, $P_1 - RT_0$.

Table 5.7
Effectivity statistics, adapted mesh, rough solution, $P_2 - RT_0$.

Max. eff	Min. eff	Mean eff	Median eff	Std. deviation
1.154280	0.134993	0.999026	1.000025	0.038547

Table 5.8
Effectivity statistics, rough adaptive solution, $P_1 - RT_0$.

Max. eff	Min. eff	Mean eff	Median eff	Std. deviation
5.098134	0.491717	1.343084	1.305963	0.248348

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