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FIRST-ORDER SYSTEM LEAST SQUARES FOR THE STOKES EQUATIONS, WITH APPLICATION TO LINEAR ELASTICITY*

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Abstract. Following our earlier work on general second-order scalar equations, here we develop a least-squares functional for the two- and three-dimensional Stokes equations, generalized slightly by allowing a pressure term in the continuity equation. By introducing a *velocity flux* variable and associated curl and *trace* equations, we are able to establish ellipticity in an H^1 product norm appropriately weighted by the Reynolds number. This immediately yields optimal discretization error estimates for finite element spaces in this norm and optimal algebraic convergence estimates for multiplicative and additive multigrid methods applied to the resulting discrete systems. Both estimates are naturally uniform in the Reynolds number. Moreover, our pressure-perturbed form of the generalized Stokes equations allows us to develop an analogous result for the Dirichlet problem for linear elasticity, where we obtain the more substantive result that the estimates are uniform in the Poisson ratio.

Key words. least squares, multigrid, Stokes equations

AMS subject classifications. 65F10, 65F30

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1. Introduction.

In earlier work [10], [11], we developed least-squares functionals for a first-order system formulation of general second-order elliptic scalar partial differential equations. The functional developed in [11] was shown to be elliptic in the sense that its homogeneous form applied to the $n+1$ variables (*pressure* and *velocities*) is equivalent to the $(H^1)^{n+1}$ norm. This means that the individual variables in the functional are essentially decoupled (more precisely, their interactions are essentially subdominant). This important property ensures that standard finite element methods are of H^1-optimal accuracy in each variable and that multiplicative and additive multigrid methods applied to the resulting discrete equations are optimally convergent.

The purpose of this paper is to extend this methodology to the Stokes equations in two and three dimensions. To this end, we begin by reformulating the Stokes equations as a first-order system derived in terms of an additional vector variable, the *velocity flux*, defined as the vector of gradients of the Stokes velocities. We first apply a least-squares principle to this system using L^2 and H^{-1} norms weighted appropriately by the Reynolds number Re. We then show that the resulting functional is elliptic in a product norm involving Re and the L^2 and H^1 norms. While of theoretical interest in its own right, we use this result here primarily as a vehicle for establishing that a modified form of this functional is fully elliptic in an H^1 product norm scaled by Re.

This appears to be the first general theory of this kind for the Stokes equations in general dimensions with velocity boundary conditions. Bochev and Gunzburger [6]

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developed least-squares functionals for Stokes equations in norms that include stronger Sobolev terms and mesh weighting, but none are product H^1 elliptic. Chang [12] also used velocity derivative variables to derive a product H^1 elliptic functional for Stokes equations, but it is inherently limited to two dimensions. For general dimensions, a vorticity–velocity–pressure form (cf. [4] and [16]) proved to be product H^1 elliptic but only for certain nonstandard boundary conditions. For the more practical (cf. [15], [18], and [20]) velocity boundary conditions treated here, the velocity–vorticity–pressure formulation examined by Chang [13] can be shown by counterexample [3] not to be equivalent to any H^1 product norm, even with the added boundary condition on the normal component of vorticity. Moreover, this formulation admits no apparent additional equation, such as those introduced below for our formulation, that would enable such an equivalence. The velocity–pressure–stress formulation described in [7] has the same shortcomings. (If the vorticity and deformation stress variables are important, then they can be easily and accurately reconstructed from the velocity flux variables introduced in our formulation.)

Our modified Stokes functional is obtained essentially by extending the first-order system with curl and scalar (*trace*) equations involving certain derivatives of the velocity flux variable, then appealing to a simple L^2 least-squares principle. As in [11] for the scalar case, the important H^1 ellipticity property that we establish guarantees optimal finite element accuracy and multigrid convergence rates applied to this Stokes least-squares functional that are naturally uniform in Re.

While this least-squares form requires several new dependent variables, we believe that the added cost is more than offset by the strengthened accuracy of the discretization and the speed that the attendant multigrid solution process attains. Moreover, while H^2 regularity is needed to obtain full product H^1 ellipticity, this is to be expected since it implies optimal discretization and solver estimates in all variables, including velocity fluxes. (We thus obtain optimal H^1 estimates for the derivatives of velocity.) In any case, strengthened regularity is not necessary for this functional if we do not insist on full product H^1 norm equivalence, and it is not at all necessary for the H^{-1} functionals we introduce.

On the other hand, a possible real limitation of our L^2 approach is that it applies only in the common case that the source terms are in L^2. For completeness, we therefore introduce an additional H^{-1}-type functional that applies to the extended system with the curl and trace equations. This approach is somewhat more complicated than the basic H^{-1} scheme applied to the original first-order system, but it is otherwise superior in the sense that it inherits most of the other practical benefits of the L^2 functional.

One of the more compelling benefits of least squares is the freedom to incorporate additional equations and impose additional boundary conditions as long as the system is consistent. In fact, many problems are perhaps best treated with overdetermined (but consistent) first-order systems, as we have here for Stokes. We therefore abandon the so-called Agmon–Douglas–Nirenberg (ADN) theory (cf. [1], [2]), which is restricted to square systems, in favor of more direct tools of analysis.

An important aspect of our general formulation is that it applies equally well to the Dirichlet problem for linear elasticity. This is done by posing the Stokes equations in a slightly generalized form that includes a pressure term in the continuity equation. Our development and results then automatically apply to linear elasticity. Most importantly, our optimal discretization and solver estimates are uniform in the Poisson ratio.

We emphasize that the discretization and algebraic convergence properties of our L^2 functional for the generalized Stokes equations are automatic consequences of the

H^1 product norm ellipticity established here and the finite element and multigrid theories established in sections 3–5 of [11]. We are therefore content with an abbreviated paper that focuses on establishing ellipticity, which we do in section 3. Section 2 introduces the generalized Stokes equations, the two relevant first-order systems and their functionals, and some preliminary theory. Concluding remarks are made in section 4.

2. The Stokes problem, its first-order system formulation, and other preliminaries. Let Ω be a bounded, open, connected domain in $\Re^n$ ($n = 2$ or 3) with Lipschitz boundary $\partial\Omega$. The pressure-perturbed form of the generalized stationary Stokes equations in dimensionless variables may be written as

$$\left\{ \begin{array}{rcll} -\nu\Delta\,\mathbf{u} + \nabla\,p &=& \mathbf{f} & \text{in} \quad \Omega, \\ \nabla\cdot\,\mathbf{u} + \delta\,p &=& g & \text{in} \quad \Omega, \end{array} \right. \tag{2.1}$$

where the symbols Δ, ∇, and $\nabla\cdot$ stand for the Laplacian, gradient, and divergence operators, respectively ($\Delta\,\mathbf{u}$ signifies the n-vector of components $\Delta\,u_i$, that is, Δ applies to $\mathbf{u}$ componentwise); ν is the reciprocal of the Reynolds number Re; $\mathbf{f}$ is a given vector function; g is a given scalar function; and δ is a fixed nonnegative constant ($\delta = 0$ for Stokes, $\delta = 1$ for linear elasticity, and δ is assumed to be bounded uniformly in ν for the general case). Without loss of generality, we may assume that

$$\int_\Omega g\,dz = \int_\Omega p\,dz = 0. \tag{2.2}$$

(For $\delta = 0$, (2.1) can have a solution only when g satisfies (2.2), and we are then free to ask that p satisfy (2.2). For $\delta > 0$, in general, we have only that $\int_\Omega g\,dz = \delta\int_\Omega p\,dz$, but this can be reduced to (2.2) simply by replacing p by $p - \frac{g}{\delta}$ and g by zero in (2.1).) We consider the (generalized) Stokes equations (2.1) together with the Dirichlet velocity boundary condition

$$\mathbf{u} = \mathbf{0} \quad \text{on } \partial\Omega. \tag{2.3}$$

The slightly generalized Stokes equations in (2.1) allow our results to apply to linear elasticity. In particular, consider the Dirichlet problem

$$\left\{ \begin{array}{rcll} -\mu\Delta\,\mathbf{u} - (\lambda+\mu)\nabla\nabla\cdot\,\mathbf{u} &=& \mathbf{f} & \text{in} \quad \Omega, \\ \mathbf{u} &=& \mathbf{0} & \text{on} \quad \partial\Omega, \end{array} \right. \tag{2.4}$$

where $\mathbf{u}$ now represents displacements and $\mu > 0$ and $\lambda > -2M/3$ are the Lamé constants. This is recast in form (2.1)–(2.2) by introducing the pressure variable[1] $p = -\nabla\cdot\,\mathbf{u}$, by rescaling $\mathbf{f}$, and by letting $g = 0$, $\delta = 1$, and[2] $\nu = \frac{\mu}{\lambda+\mu}$. (It is easy to see that this p must satisfy (2.2).) An important consequence of the results we develop below is that standard Rayleigh–Ritz discretization and multigrid solution methods can be applied with optimal estimates that are uniform in the mesh size and Poisson ratio. For example, we obtain optimal uniform approximation of the gradients of displacements in the H^1 product norm. This in turn implies analogous

[1]Perhaps a more physical choice for this artificial pressure would have been $p = -\frac{\lambda}{2\mu}\,\nabla\cdot\,\mathbf{u}$, since it then becomes the hydrostatic pressure in the incompressible limit. We chose our particular scaling because it most easily conforms to (2.1). In any case, our results apply to virtually any nonnegative scaling of p, with no effect on the equivalence constants (provided the norms are correspondingly scaled); see Theorems 3.1 and 3.2.

[2]The Greek symbol ν used in fluid dynamics for the inverse of the Reynolds number is unfortunately also used in elasticity for the *Poisson ratio* $\frac{\lambda}{2(\lambda+\mu)}$. To avoid confusion, we use ν here always in the fluid dynamics sense and refer to the Poisson ratio explicitly when necessary.

H^1 estimates for the stresses, which are easily obtained from the "velocity fluxes". For related results with a different methodology and weaker norm estimates, see [14].

Let $\mathbf{curl} \equiv \nabla\times$ denote the curl operator. Here and henceforth we use notation for the case $n = 3$ and consider the special case $n = 2$ in the natural way by identifying $\Re^2$ with the (x_1, x_2)-plane in $\Re^3$. Thus, if $\mathbf{u}$ is two dimensional, then the curl of $\mathbf{u}$ means the scalar function

$$\nabla\times \mathbf{u} = \partial_1 u_2 - \partial_2 u_1,$$

where u_1 and u_2 are the components of $\mathbf{u}$. The following identity is immediate:

$$\nabla\times (\nabla\times \mathbf{u}) = -\Delta\, \mathbf{u} + \nabla\, (\nabla\cdot\, \mathbf{u}). \tag{2.5}$$

For $n = 2$, (2.5) is interpreted as

$$\nabla^{\perp}\, (\nabla\times \mathbf{u}) = -\Delta\, \mathbf{u} + \nabla\, (\nabla\cdot\, \mathbf{u}),$$

where $\nabla^{\perp}$ is the formal adjoint of $\nabla\times$ defined by

$$\nabla^{\perp} q = \begin{pmatrix} \partial_2 q \\ -\partial_1 q \end{pmatrix}.$$

We will be introducing a new independent variable defined as the n^2-vector function of gradients of the u_i, $i = 1, 2, ..., n$. It will be convenient to view the original n-vector functions as column vectors and the new n^2-vector functions as either block column vectors or matrices. Thus, given

$$\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}$$

and denoting $\mathbf{u}^t = (u_1, u_2, ..., u_n)$, then an operator G defined on scalar functions (e.g., $G = \nabla$) is extended to n-vectors componentwise:

$$G\mathbf{u}^t = (Gu_1, Gu_2, ..., Gu_n)$$

and

$$G\mathbf{u} = \begin{pmatrix} Gu_1 \\ Gu_2 \\ \vdots \\ Gu_n \end{pmatrix}.$$

If $\mathbf{U}_i \equiv G\mathbf{u}_i$ is a n-vector function, then we write the matrix

$$\begin{aligned} \underline{\mathbf{U}} \equiv G\mathbf{u}^t &= (\mathbf{U}_1, \mathbf{U}_2, ..., \mathbf{U}_n) \\ &= \begin{pmatrix} U_{11} & U_{12} & \cdots & U_{1n} \\ U_{21} & U_{22} & \cdots & U_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ U_{n1} & U_{n2} & \cdots & U_{nn} \end{pmatrix}. \end{aligned}$$

We then define the trace operator tr according to

$$\mathrm{tr}\underline{\mathbf{U}} = \sum_{i=1}^{n} U_{ii}.$$

If D is an operator on n-vector functions (e.g., $D = \nabla\times$), then its extension to matrices is defined by

$$D\underline{\mathbf{U}} = (D\mathbf{U}_1,\, D\mathbf{U}_2, \ldots,\, D\mathbf{U}_n).$$

When each $D\mathbf{U}_i$ is a scalar function (e.g., $D = \nabla\cdot$), then we will want to view the extension as a mapping to column vectors, so we will use the convention

$$(D\underline{\mathbf{U}})^t = \begin{pmatrix} D\mathbf{U}_1 \\ D\mathbf{U}_2 \\ \vdots \\ D\mathbf{U}_n \end{pmatrix}.$$

We also extend the tangential operator $\mathbf{n}\times$ componentwise ($\mathbf{n}$ denotes the outward unit normal on $\partial\Omega$):

$$\mathbf{n}\times\underline{\mathbf{U}} = (\mathbf{n}\times\mathbf{U}_1,\, \mathbf{n}\times\mathbf{U}_2, \ldots,\, \mathbf{n}\times\mathbf{U}_n).$$

Finally, inner products and norms on the matrix functions are defined in the natural componentwise way, e.g.,

$$\|\underline{\mathbf{U}}\|^2 = \sum_{i=1}^{n} \|\mathbf{U}_i\|^2 = \sum_{i,\,j=1}^{n} \|U_{ij}\|^2.$$

Introducing the *velocity flux* variable

$$\underline{\mathbf{U}} = \nabla\mathbf{u}^t = (\nabla\, u_1,\, \nabla\, u_2, \ldots,\, \nabla\, u_n),$$

then the Stokes system (2.1) and (2.3) may be recast as the following equivalent first-order system:

$$\left\{\begin{array}{rcll} \underline{\mathbf{U}} - \nabla\,\mathbf{u}^t &=& \underline{\mathbf{0}} & \text{in} \quad \Omega, \\ -\nu\,(\nabla\cdot\underline{\mathbf{U}})^t + \nabla\, p &=& \mathbf{f} & \text{in} \quad \Omega, \\ \nabla\cdot\mathbf{u} + \delta\, p &=& g & \text{in} \quad \Omega, \\ \mathbf{u} &=& \mathbf{0} & \text{on} \quad \partial\Omega. \end{array}\right. \tag{2.6}$$

Note that the definition of $\underline{\mathbf{U}}$, the "continuity" condition $\nabla\cdot\mathbf{u} + \delta\, p = g$ in Ω, and the Dirichlet condition $\mathbf{u} = \mathbf{0}$ on $\partial\Omega$ imply the respective properties

$$\nabla\times\underline{\mathbf{U}} = \underline{\mathbf{0}} \quad \text{in } \Omega, \quad \operatorname{tr}\underline{\mathbf{U}} + \delta\, p = g \quad \text{in } \Omega, \quad \text{and } \mathbf{n}\times\underline{\mathbf{U}} = \underline{\mathbf{0}} \quad \text{on } \partial\Omega. \tag{2.7}$$

Then an equivalent extended system for (2.6) is

$$\left\{\begin{array}{rcll} \underline{\mathbf{U}} - \nabla\,\mathbf{u}^t &=& \mathbf{0} & \text{in} \quad \Omega, \\ -\nu\,(\nabla\cdot\underline{\mathbf{U}})^t + \nabla\, p &=& \mathbf{f} & \text{in} \quad \Omega, \\ \nabla\cdot\mathbf{u} + \delta\, p &=& g & \text{in} \quad \Omega, \\ \nabla\operatorname{tr}\underline{\mathbf{U}} + \delta\nabla\, p &=& \nabla g & \text{in} \quad \Omega, \\ \nabla\times\underline{\mathbf{U}} &=& \underline{\mathbf{0}} & \text{in} \quad \Omega, \\ \mathbf{u} &=& \mathbf{0} & \text{on} \quad \partial\Omega, \\ \mathbf{n}\times\underline{\mathbf{U}} &=& \underline{\mathbf{0}} & \text{on} \quad \partial\Omega. \end{array}\right. \tag{2.8}$$

Let $\mathcal{D}(\Omega)$ be the linear space of infinitely differentiable functions with compact support on Ω and let $\mathcal{D}'(\Omega)$ denote the dual space of $\mathcal{D}(\Omega)$. The duality pairing between $\mathcal{D}'(\Omega)$ and $\mathcal{D}(\Omega)$ is denoted by $\langle\cdot,\,\cdot\rangle$. We use the standard notation and definition for the Sobolev spaces $H^s(\Omega)^n$ and $H^s(\partial\Omega)^n$ for $s \geq 0$; the standard associated

inner products are denoted by $(\cdot,\,\cdot)_{s,\,\Omega}$ and $(\cdot,\,\cdot)_{s,\,\partial\Omega}$, and their respective norms are denoted by $\|\cdot\|_{s,\,\Omega}$ and $\|\cdot\|_{s,\,\partial\Omega}$. (We suppress the superscript n because dependence of the vector norms on dimension will be clear by context. We also omit Ω from the inner product and norm designation when there is no risk of confusion.) For $s=0$, $H^s(\Omega)^n$ coincides with $L^2(\Omega)^n$. In this case, the norm and inner product will be denoted by $\|\cdot\|$ and $(\cdot,\,\cdot)$, respectively. As usual, $H_0^s(\Omega)$ is the closure of $\mathcal{D}(\Omega)$ with respect to the norm $\|\cdot\|_s$ and $H^{-s}(\Omega)$ is its dual with norm defined by

$$\|\varphi\|_{-s} = \sup_{0\neq\phi\in H_0^s(\Omega)} \frac{\langle\varphi,\,\phi\rangle}{\|\phi\|_s}.$$

Define the product spaces $H_0^s(\Omega)^n = \prod_{i=1}^n H_0^s(\Omega)$ and $H^{-s}(\Omega)^n = \prod_{i=1}^n H^{-s}(\Omega)$ with standard product norms. Let

$$H(\mathrm{div};\,\Omega) = \{\mathbf{v}\in L^2(\Omega)^n \,:\, \nabla\cdot\mathbf{v}\in L^2(\Omega)\}$$

and

$$H(\mathbf{curl};\,\Omega) = \{\mathbf{v}\in L^2(\Omega)^n \,:\, \nabla\times\mathbf{v}\in L^2(\Omega)^{2n-3}\},$$

which are Hilbert spaces under the respective norms

$$\|\mathbf{v}\|_{H(\mathrm{div};\,\Omega)} \equiv \left(\|\mathbf{v}\|^2 + \|\nabla\cdot\,\mathbf{v}\|^2\right)^{\frac{1}{2}}$$

and

$$\|\mathbf{v}\|_{H(\mathbf{curl};\,\Omega)} \equiv \left(\|\mathbf{v}\|^2 + \|\nabla\times\mathbf{v}\|^2\right)^{\frac{1}{2}}.$$

Define their subspaces

$$H_0(\mathrm{div};\,\Omega) = \{\mathbf{v}\in H(\mathrm{div};\,\Omega) \,:\, \mathbf{n}\cdot\mathbf{v}=0 \text{ on } \partial\Omega\}$$

and

$$H_0(\mathbf{curl};\,\Omega) = \{\mathbf{v}\in H(\mathbf{curl};\,\Omega) \,:\, \mathbf{n}\times\mathbf{v}=\mathbf{0} \text{ on } \partial\Omega\}.$$

Finally, define

$$L_0^2(\Omega)^n = \left\{\mathbf{v}\in L^2(\Omega)^n \,:\, \int_\Omega v_i\,dz = 0 \text{ for } i=1,...,n\right\}.$$

It is well known that the (weak form of the) boundary value problem (2.1)–(2.2) has a unique solution $(\mathbf{u},\,p)\in H_0^1(\Omega)^n\times L_0^2(\Omega)$ for any $\mathbf{f}\in H^{-1}(\Omega)^n$ and for $g\in H^1(\Omega)$ (e.g., see [17], [18], [15]). Moreover, if the boundary of the domain Ω is $C^{1,\,1}$, then the following H^2 regularity result holds:

$$\|\nu\,\mathbf{u}\|_2 + \|p\|_1 \le C\,\left(\|\mathbf{f}\| + \|\nu\,g\|_1\right). \tag{2.9}$$

(We use C with or without subscripts in this paper to denote a generic positive constant, possibly different at different occurrences, that is independent of the Reynolds number and other parameters introduced in this paper but may depend on the domain Ω or the constant δ.) Bound (2.9) is established for the case $\nu=1$ and $\delta=0$ in [18]; the case for general ν and $\delta=0$ is then immediate; and the case $\delta>0$ follows from the well known (cf. [9]) linear elasticity bound $\|\mathbf{u}\|_2 + \|\lambda\nabla\cdot\mathbf{u}\|_1 \le C\,\|\mathbf{f}\|$, where $\mathbf{f}$ is the (unscaled) source term in (2.4). When the domain Ω is a convex polygon, bound

(2.9) is still valid when $\delta \geq 0$. For the case $\delta = 0$, we have only the weaker estimate established in [17]:

$$\|\nu\,\mathbf{u}\|_2 + \|p\|_1 \leq C\,\left(\|\mathbf{f}\| + \|\nu d^{-1}\,g\| + |\nu\,g|_1\right), \tag{2.10}$$

where $d = d(x)$ denotes the distance from interior point $x \in \Omega$ to a closest vertex of $\partial\Omega$. We will need (2.9) and (2.10) to establish full H^1 product ellipticity of one of our reformulations of (2.1)–(2.2) (see Theorem 3.2 and Corollary 3.1).

The following lemma is an immediate consequence of a general functional analysis result due to Nečas [19] (see also [15]).

LEMMA 2.1. *For any p in $L_0^2(\Omega)$ we have*

$$\|p\| \leq C\,\|\nabla\,p\|_{-1}. \tag{2.11}$$

Proof. See [19] for a general proof. □

A curl result analogous to Green's theorem for divergence follows from [15, Chapter I, Theorem 2.11]:

$$(\nabla\times\mathbf{z},\ \boldsymbol{\phi}) = (\mathbf{z},\ \nabla\times\boldsymbol{\phi}) - \int_{\partial\Omega} \boldsymbol{\phi}\cdot(\mathbf{n}\times\mathbf{z})\,ds \tag{2.12}$$

for $\mathbf{z} \in H(\mathbf{curl}\,;\,\Omega)$ and $\boldsymbol{\phi} \in H^1(\Omega)^n$.

Finally, we summarize results from [15] that we will need for G_2 in the next section. The first inequality follows from Theorems 3.7–3.9 in [15], while the second inequality follows from Lemmas 3.4 and 3.6 in [15].

THEOREM 2.1. *Assume that the domain Ω is a bounded convex polyhedron or has $C^{1,1}$ boundary. Then for any vector function $\mathbf{v}$ in either $H_0(\mathrm{div};\,\Omega)\cap H(\mathbf{curl};\,\Omega)$ or $H(\mathrm{div};\,\Omega)\cap H_0(\mathbf{curl};\,\Omega)$ we have*

$$\|\mathbf{v}\|_1^2 \leq C\,\left(\|\mathbf{v}\|^2 + \|\nabla\cdot\,\mathbf{v}\|^2 + \|\nabla\times\mathbf{v}\|^2\right). \tag{2.13}$$

If, in addition, the domain is simply connected, then

$$\|\mathbf{v}\|_1^2 \leq C\,\left(\|\nabla\cdot\,\mathbf{v}\|^2 + \|\nabla\times\mathbf{v}\|^2\right). \tag{2.14}$$

3. First-order system least squares. In this section, we consider least-squares functionals based on system (2.6) and its extension (2.8). Our primary objective here is to establish ellipticity of these least-squares functionals in the appropriate Sobolev spaces.

Our first least-squares functional is defined in terms of appropriate weights and norms of the residuals for system (2.6):

$$\begin{aligned} G_1(\underline{\mathbf{U}},\,\mathbf{u},\,p;\,\mathbf{f},\,g) = {} & \|\mathbf{f} + \nu\,(\nabla\cdot\underline{\mathbf{U}})^t - \nabla\,p\|_{-1}^2 + \nu^2\,\|\underline{\mathbf{U}} - \nabla\,\mathbf{u}^t\|^2 \\ & + \nu^2\,\|\nabla\cdot\mathbf{u} + \delta\,p - g\|^2. \end{aligned} \tag{3.1}$$

Note the use of the H^{-1} norm in the first term here. Our second functional is defined as a weighted sum of the L^2 norms of the residuals for extended system (2.8):

$$\begin{aligned} G_2(\underline{\mathbf{U}},\,\mathbf{u},\,p;\,\mathbf{f},\,g) = {} & \|\mathbf{f} + \nu\,(\nabla\cdot\underline{\mathbf{U}})^t - \nabla\,p\|^2 + \nu^2\,\|\underline{\mathbf{U}} - \nabla\,\mathbf{u}^t\|^2 + \nu^2\,\|\nabla\times\underline{\mathbf{U}}\|^2 \\ & + \nu^2\,\|\nabla\cdot\mathbf{u} + \delta\,p - g\|^2 + \nu^2\,\|\nabla\mathrm{tr}\,\underline{\mathbf{U}} + \delta\nabla\,p - \nabla g\|^2. \end{aligned} \tag{3.2}$$

Since the known H^2 regularity result for $\delta = 0$ and a convex polyhedral domain is the weaker bound (2.10), we must modify our second functional accordingly:

$$\begin{aligned} G_3(\underline{\mathbf{U}},\,\mathbf{u},\,p;\,\mathbf{f},\,g) = {} & \|\mathbf{f} + \nu\,(\nabla\cdot\underline{\mathbf{U}})^t - \nabla\,p\|^2 + \nu^2\,\|\underline{\mathbf{U}} - \nabla\,\mathbf{u}^t\|^2 + \nu^2\,\|\nabla\times\underline{\mathbf{U}}\|^2 \\ & + \nu^2\,\|\nabla\cdot\mathbf{u} - g\|^2 + \nu^2\,\|\nabla\mathrm{tr}\,\underline{\mathbf{U}} - \nabla g\|^2 + \nu^2\,\|d^{-1}(\mathrm{tr}\,\underline{\mathbf{U}} - g)\|^2. \end{aligned} \tag{3.3}$$

Let

$$\mathcal{V}_0 = \{\underline{\mathbf{V}} \in H^1(\Omega)^{n^2} \,:\, \mathbf{n} \times \underline{\mathbf{V}} = \underline{\mathbf{0}} \quad \text{on } \partial\Omega\}$$

and

$$\mathcal{V}_1 = \{\underline{\mathbf{V}} \in \mathcal{V}_0 \,:\, d^{-1}\mathrm{tr}\underline{\mathbf{V}} \in L^2(\Omega)\},$$

and define

$$\mathbf{V}_1 = L^2(\Omega)^{n^2} \times H_0^1(\Omega)^n \times L_0^2(\Omega), \quad \mathbf{V}_2 = \mathcal{V}_0 \times H_0^1(\Omega)^n \times (H^1(\Omega)/\Re),$$

and

$$\mathbf{V}_3 = \mathcal{V}_1 \times H_0^1(\Omega)^n \times (H^1(\Omega)/\Re).$$

Note that $\mathbf{V}_3 \subset \mathbf{V}_2 \subset \mathbf{V}_1$. For $i = 1, 2$, or 3, the first-order system least-squares variational problem for the Stokes equations is to minimize the quadratic functional $G_i(\underline{\mathbf{U}}, \mathbf{u}, p; \mathbf{f}, g)$ over $\mathbf{V}_i$: find $(\underline{\mathbf{U}}, \mathbf{u}, p) \in \mathbf{V}_i$ such that

$$G_i(\underline{\mathbf{U}}, \mathbf{u}, p; \mathbf{f}, g) = \inf_{(\underline{\mathbf{V}}, \mathbf{v}, q) \in \mathbf{V}_i} G_i(\underline{\mathbf{V}}, \mathbf{v}, q; \mathbf{f}, g). \tag{3.4}$$

THEOREM 3.1. *There exists a constant C independent of ν such that for any $(\underline{\mathbf{U}}, \mathbf{u}, p) \in \mathbf{V}_1$ we have*

$$\frac{1}{C}\left(\nu^2\|\underline{\mathbf{U}}\|^2 + \nu^2\|\mathbf{u}\|_1^2 + \|p\|^2\right) \le G_1(\underline{\mathbf{U}}, \mathbf{u}, p; \mathbf{0}, 0) \tag{3.5}$$

and

$$G_1(\underline{\mathbf{U}}, \mathbf{u}, p; \mathbf{0}, 0) \le C\left(\nu^2\|\underline{\mathbf{U}}\|^2 + \nu^2\|\mathbf{u}\|_1^2 + \|p\|^2\right). \tag{3.6}$$

Proof. Upper bound (3.6) is straightforward from the triangle and Cauchy–Schwarz inequalities. We proceed to show the validity of (3.5) for $(\underline{\mathbf{U}}, \mathbf{u}, p) \in \mathbf{W}_1 \equiv H(\mathrm{div};\Omega)^n \times H_0^1(\Omega)^n \times (L_0^2(\Omega) \cap H^1(\Omega))$. Then (3.5) would follow for $(\underline{\mathbf{U}}, \mathbf{u}, p) \in \mathbf{V}_1$ by continuity. For any $(\underline{\mathbf{U}}, \mathbf{u}, p) \in \mathbf{W}_1$ and $\boldsymbol{\phi} \in H_0^1(\Omega)^n$, we have

$$\begin{aligned}(\nabla p, \boldsymbol{\phi}) &= (-\nu\,(\nabla\cdot\underline{\mathbf{U}})^t + \nabla p, \boldsymbol{\phi}) - \nu\,(\underline{\mathbf{U}}, \nabla\boldsymbol{\phi}^t)\\ &\le \|-\nu\,(\nabla\cdot\underline{\mathbf{U}})^t + \nabla p\|_{-1}\|\boldsymbol{\phi}\|_1 + \nu\,\|\underline{\mathbf{U}}\|\,\|\nabla\boldsymbol{\phi}^t\|.\end{aligned}$$

Hence, by Lemma 2.1, we have

$$\|p\| \le C\,\left(\|-\nu\,(\nabla\cdot\underline{\mathbf{U}})^t + \nabla p\|_{-1} + \nu\,\|\underline{\mathbf{U}}\|\right). \tag{3.7}$$

From (3.7) and the Poincaré–Friedrichs inequality on $\mathbf{u}$ we have

$$\begin{aligned}&\nu^2\|\nabla\mathbf{u}^t\|^2\\ &\quad= \nu^2\,(\nabla\mathbf{u}^t - \underline{\mathbf{U}}, \nabla\mathbf{u}^t) + \nu\,(-\nu\,(\nabla\cdot\underline{\mathbf{U}})^t + \nabla p, \mathbf{u}) + \nu\,(p, \nabla\cdot\mathbf{u} + \delta\,p) - \nu\delta(p,p)\\ &\quad\le \nu^2\,\|\nabla\mathbf{u}^t - \underline{\mathbf{U}}\|\,\|\nabla\mathbf{u}^t\| + \nu\,\|-\nu\,(\nabla\cdot\underline{\mathbf{U}})^t + \nabla p\|_{-1}\|\mathbf{u}\|_1 + \nu\,\|p\|\,\|\nabla\cdot\mathbf{u} + \delta\,p\|\\ &\quad\le (\nu\,\|\nabla\mathbf{u}^t - \underline{\mathbf{U}}\| + C\,\|-\nu\,(\nabla\cdot\underline{\mathbf{U}})^t + \nabla p\|_{-1})\;\nu\|\nabla\mathbf{u}^t\|\\ &\qquad + C\|-\nu\,(\nabla\cdot\underline{\mathbf{U}})^t + \nabla p\|_{-1}\nu\,\|\nabla\cdot\mathbf{u} + \delta\,p\| + C\,\nu^2\|\underline{\mathbf{U}}\|\,\|\nabla\cdot\mathbf{u} + \delta\,p\|.\end{aligned}$$

Using the ε-inequality $2ab \le \frac{1}{\varepsilon}a^2 + \varepsilon b^2$ with $\varepsilon = 1$ for the first two products yields

$$\nu^2\|\nabla \mathbf{u}^t\|^2 \le C\, G_1(\underline{\mathbf{U}},\, \mathbf{u},\, p;\, \mathbf{0},\, 0) + C\,\nu^2\|\underline{\mathbf{U}}\|\,\|\nabla\cdot\mathbf{u} + \delta\, p\|. \tag{3.8}$$

Again from (3.7) and the Poincaré–Friedrichs inequality on $\mathbf{u}$ we have

$$\begin{aligned}
&\nu^2\|\underline{\mathbf{U}}\|^2\\
&\quad = \nu^2\,(\underline{\mathbf{U}} - \nabla\,\mathbf{u}^t,\, \underline{\mathbf{U}}) + \nu\,(\mathbf{u},\, -\nu\,(\nabla\cdot\underline{\mathbf{U}})^t + \nabla\, p) + \nu\,(\nabla\cdot\mathbf{u} + \delta\, p,\, p) - \nu\delta(p,p)\\
&\quad \le \nu^2\,\|\underline{\mathbf{U}} - \nabla\,\mathbf{u}^t\|\,\|\underline{\mathbf{U}}\| + C\,\nu\,\|\nabla\,\mathbf{u}^t\|\,\|-\nu\,(\nabla\cdot\underline{\mathbf{U}})^t + \nabla\, p\|_{-1} + \nu\,\|p\|\,\|\nabla\cdot\mathbf{u} + \delta\, p\|\\
&\quad \le \nu^2\,\|\underline{\mathbf{U}} - \nabla\,\mathbf{u}^t\|\,\|\underline{\mathbf{U}}\| + C\,\nu\,\|\nabla\,\mathbf{u}^t\|\,\|-\nu\,(\nabla\cdot\underline{\mathbf{U}})^t + \nabla\, p\|_{-1}\\
&\qquad + C\|-\nu\,(\nabla\cdot\underline{\mathbf{U}})^t + \nabla\, p\|_{-1}\nu\,\|\nabla\cdot\mathbf{u} + \delta\, p\| + C\,\nu^2\|\underline{\mathbf{U}}\|\,\|\nabla\cdot\mathbf{u} + \delta\, p\|.
\end{aligned}$$

Using the ε-inequality on the first three products and (3.8), we then have

$$\begin{aligned}
\nu^2\|\underline{\mathbf{U}}\|^2 &\le C\, G_1(\underline{\mathbf{U}},\, \mathbf{u},\, p;\, \mathbf{0},\, 0) + C\,\nu^2\,\|\nabla\,\mathbf{u}^t\|^2 + C\,\nu^2\,\|\underline{\mathbf{U}}\|\,\|\nabla\cdot\mathbf{u} + \delta\, p\|\\
&\le C\, G_1(\underline{\mathbf{U}},\, \mathbf{u},\, p;\, \mathbf{0},\, 0) + C\,\nu^2\,\|\underline{\mathbf{U}}\|\,\|\nabla\cdot\mathbf{u} + \delta\, p\|.
\end{aligned}$$

Again using the ε-inequality we find that

$$\nu^2\|\underline{\mathbf{U}}\|^2 \le C\, G_1(\underline{\mathbf{U}},\, \mathbf{u},\, p;\, \mathbf{0},\, 0). \tag{3.9}$$

Using (3.9) in (3.7) and (3.8), we now have that

$$\|p\|^2 \le C\, G_1(\underline{\mathbf{U}},\, \mathbf{u},\, p;\, \mathbf{0},\, 0) \quad \text{and} \quad \nu^2\|\nabla\,\mathbf{u}^t\|^2 \le C\, G_1(\underline{\mathbf{U}},\, \mathbf{u},\, p;\, \mathbf{0},\, 0).$$

The theorem now follows from these bounds, (3.9), and the Poincaré–Friedrichs inequality on $\mathbf{u}$. ∎

The next two lemmas will be useful in the proof of Theorem 3.2.

LEMMA 3.1 (Poincaré–Friedrichs-type inequality). *Suppose that the assumptions of Theorem* 2.1 *hold. Let* $p \in H^1(\Omega)$ *satisfy* $\int_\Omega p\, dz = 0$*; then*

$$\|p\| \le C|p|_1, \tag{3.10}$$

where C *depends only on* Ω*. Further, let* $\mathbf{q} \in (H_0^1(\Omega) \cap H^2(\Omega))^n$*; then*

$$\|\nabla\cdot\mathbf{q} + \delta p\| \le C|\nabla\cdot\mathbf{q} + \delta p|_1, \tag{3.11}$$

where C *depends only on* Ω*.*

Proof. The equation $\int_\Omega p\, dz = 0$ implies $p = 0$ at some point in Ω. The first result now follows from the standard Poincaré–Friedrichs inequality. The second result follows from the fact that $\int_\Omega(\nabla\cdot\mathbf{q} + \delta p)\, dz = 0$. ∎

LEMMA 3.2. *Under the assumptions of Theorem* 2.1 *with simply connected* Ω*, for any* p *in* $H^1(\Omega)$ *we have the following:*

($n = 2$) *let* $\boldsymbol{\phi} = (\phi_1,\, \phi_2)^t$ *and* $\mathbf{q} = (q_1,\, q_2)^t$*; if each* $q_i \in H_0^1(\Omega) \cap H^2(\Omega)$ *and each* $\phi_i \in H^1(\Omega)$ *is such that* $\Delta\phi_i \in L^2(\Omega)$ *and* $\mathbf{n}\cdot\nabla\,\phi_i = 0$ *on* $\partial\Omega$*, then*

$$|\nabla\cdot\mathbf{q} + \delta\, p|_1^2 \le C\,\left(|\nabla\cdot\mathbf{q} + \operatorname{tr}\nabla^\perp\boldsymbol{\phi}^t + \delta\, p|_1^2 + \|\Delta\boldsymbol{\phi}\|^2\right); \tag{3.12}$$

($n = 3$) *let* $\underline{\Phi} = (\boldsymbol{\phi}_1,\, \boldsymbol{\phi}_2,\, \boldsymbol{\phi}_3)$ *and* $\mathbf{q} = (q_1,\, q_2,\, q_3)^t$*; if each* $q_i \in H_0^1(\Omega) \cap H^2(\Omega)$ *and each* $\boldsymbol{\phi}_i \in H^1(\Omega)^3$ *is divergence free with* $\Delta\boldsymbol{\phi}_i \in L^2(\Omega)^n$ *and* $\mathbf{n}\times(\nabla\times\boldsymbol{\phi}_i) = \mathbf{0}$ *on* $\partial\Omega$*, then*

$$|\nabla\cdot\mathbf{q} + \delta\, p|_1^2 \le C\,\left(|\nabla\cdot\mathbf{q} + \operatorname{tr}\nabla\times\underline{\Phi} + \delta\, p|_1^2 + \|\Delta\underline{\Phi}\|^2\right). \tag{3.13}$$

Proof. ($n = 2$) The assumptions of Theorem 2.1 are sufficient to guarantee H^2 regularity of the Laplace equation on Ω, that is, the second inequality in the equation

$$|\nabla\times\boldsymbol{\phi}|_1 \le C\,|\boldsymbol{\phi}|_2 \le C\,\|\Delta\boldsymbol{\phi}\|.$$

Note that $\operatorname{tr}(\nabla^\perp\phi_1,\,\nabla^\perp\phi_2) = \nabla\times\boldsymbol{\phi}$. Then, from the above and the triangle inequality, we have

$$\begin{aligned}|\nabla\cdot\mathbf{q}+\delta p|_1^2 &\le 2\left(|\nabla\cdot\mathbf{q}+\nabla\times\boldsymbol{\phi}+\delta p|_1^2+|\nabla\times\boldsymbol{\phi}|_1^2\right)\\ &\le C\left(|\nabla\cdot\mathbf{q}+\operatorname{tr}\nabla^\perp\boldsymbol{\phi}^t+\delta p|_1^2+\|\Delta\boldsymbol{\phi}\|^2\right),\end{aligned}$$

which is (3.12).

($n = 3$) Bound (2.14) with $\mathbf{v} = \nabla\times\underline{\Phi}$ and identity (2.5) applied to each column of $\nabla\times\underline{\Phi}$ imply that

$$|\operatorname{tr}\nabla\times\underline{\Phi}|_1^2 \le 3\,|\nabla\times\underline{\Phi}|_1^2 \le C\,\left(\|\nabla\cdot\nabla\times\underline{\Phi}\|^2+\|\nabla\times\nabla\times\underline{\Phi}\|^2\right) = C\,\|\Delta\underline{\Phi}\|^2$$

since each $\boldsymbol{\phi}_i$ is divergence free. Equation (3.13) now follows from the triangle inequality as for the case $n = 2$. ∎

THEOREM 3.2. *Assume that the domain Ω is a bounded convex polyhedron or has $C^{1,1}$ boundary and that regularity bound* (2.9) *holds. Then there exists a constant C independent of ν such that for any $(\underline{\mathbf{U}},\,\mathbf{u},\,p)\in\mathbf{V}_2$ we have*

$$\frac{1}{C}\left(\nu^2\,\|\underline{\mathbf{U}}\|_1^2+\nu^2\,\|\mathbf{u}\|_1^2+\|p\|_1^2\right) \le G_2(\underline{\mathbf{U}},\,\mathbf{u},\,p;\,\mathbf{0},\,0) \tag{3.14}$$

and

$$G_2(\underline{\mathbf{U}},\,\mathbf{u},\,p;\,\mathbf{0},\,0) \le C\left(\nu^2\,\|\underline{\mathbf{U}}\|_1^2+\nu^2\,\|\mathbf{u}\|_1^2+\|p\|_1^2\right). \tag{3.15}$$

Proof. Upper bound (3.15) is straightforward from the triangle and Cauchy–Schwarz inequalities. To prove (3.14), note that the H^{-1} norm of a function is always bounded by its L^2 norm. Since $\mathbf{V}_2\subset\mathbf{V}_1$, then $G_1\le G_2$ on $\mathbf{V}_2$. Hence, by Theorem 3.1, we have

$$\nu^2\|\underline{\mathbf{U}}\|^2+\nu^2\,\|\mathbf{u}\|_1^2+\|p\|^2 \le CG_1(\underline{\mathbf{U}},\,\mathbf{u},\,p;\,\mathbf{0},\,0) \le CG_2(\underline{\mathbf{U}},\,\mathbf{u},\,p;\,\mathbf{0},\,0). \tag{3.16}$$

From Theorem 2.1 and (3.10) we have

$$\nu^2\,\|\underline{\mathbf{U}}\|_1^2+\|p\|_1^2 \le C\left(\nu^2\,\|\underline{\mathbf{U}}\|^2+\nu^2\,\|(\nabla\cdot\underline{\mathbf{U}})^t\|^2+\nu^2\,\|\nabla\times\underline{\mathbf{U}}\|^2+\|\nabla\,p\|^2\right). \tag{3.17}$$

It thus suffices to show that

$$\begin{aligned}&C\left(\nu^2\,\|(\nabla\cdot\underline{\mathbf{U}})^t\|^2+\|\nabla\,p\|^2\right)\\ &\qquad\le\|-\nu(\nabla\cdot\underline{\mathbf{U}})^t+\nabla\,p\|^2+\nu^2\,|\operatorname{tr}\underline{\mathbf{U}}+\delta p|_1^2+\nu^2\,\|\nabla\times\underline{\mathbf{U}}\|^2.\end{aligned} \tag{3.18}$$

We will prove (3.18) only for the case $n = 3$ because the proof for $n = 2$ is similar. First, we assume that the domain Ω is simply connected with connected boundary. Since $\mathbf{n}\times\underline{\mathbf{U}} = \underline{\mathbf{0}}$ on $\partial\Omega$, the following decomposition is admitted:

$$\underline{\mathbf{U}} = \nabla\,\mathbf{q}^t+\nabla\times\underline{\Phi}, \tag{3.19}$$

where $\mathbf{q} \in H_0^1(\Omega)^n \cap H^2(\Omega)^n$ and $\underline{\Phi}$ is columnwise divergence free with $\mathbf{n} \times (\nabla \times \underline{\Phi}) = \underline{\mathbf{0}}$ on $\partial\Omega$. Here, we choose $\mathbf{q}$ to satisfy

$$\left\{ \begin{array}{rcll} \Delta \mathbf{q} & = & (\nabla \cdot \underline{\mathbf{U}})^t & \text{in} \quad \Omega, \\ \mathbf{q} & = & \mathbf{0} & \text{on} \quad \partial\Omega. \end{array} \right. \tag{3.20}$$

Then $\underline{\mathbf{V}} = \underline{\mathbf{U}} - \nabla \mathbf{q}^t$ is divergence free and satisfies $\mathbf{n} \times \underline{\mathbf{V}} = \mathbf{0}^t$. Since Ω has connected boundary, then $\int_{\partial\Omega} (\mathbf{n} \cdot \underline{\mathbf{V}})\, ds = \mathbf{0}^t$. Thus, Theorem 3.4 in [15] yields $\underline{\mathbf{V}} = \nabla \times \underline{\Phi}$, where $\nabla \cdot \underline{\Phi} = \mathbf{0}^t$.

By taking the curl of both sides of this decomposition, it is easy to see that

$$\|\Delta\, \underline{\Phi}\| = \|\nabla \times \underline{\mathbf{U}}\| \leq \|\underline{\mathbf{U}}\|_1, \tag{3.21}$$

so that $\|\Delta\, \underline{\Phi}\|$ is bounded and Lemma 3.2 applies. Hence,

$$\begin{aligned} &\| -\nu\, (\nabla \cdot \underline{\mathbf{U}})^t + \nabla\, p\|^2 + \nu^2\, |\mathrm{tr}\, \underline{\mathbf{U}} + \delta\, p|_1^2 + \nu^2\, \|\nabla \times \underline{\mathbf{U}}\|^2 \\ &\qquad \text{(by (3.19))} \\ &\quad = \| -\nu \Delta\, \mathbf{q} + \nabla\, p\|^2 + \nu^2\, |\nabla \cdot \mathbf{q} + \mathrm{tr}\, \nabla \times \underline{\Phi} + \delta\, p|_1^2 + \nu^2\, \|\Delta\, \underline{\Phi}\|^2, \\ &\qquad \text{(by Lemma 3.2)} \\ &\quad \geq \| -\nu \Delta\, \mathbf{q} + \nabla\, p\|^2 + C\nu^2\, |\nabla \cdot \mathbf{q} + \delta\, p|_1^2, \\ &\qquad \text{(by (3.11))} \\ &\quad \geq \| -\nu \Delta\, \mathbf{q} + \nabla\, p\|^2 + C\nu^2\, \|\nabla \cdot \mathbf{q} + \delta\, p\|_1^2, \\ &\qquad \text{(by regularity assumption (2.9) with } \mathbf{u} = \mathbf{q}) \\ &\quad \geq C\, \left(\nu^2\, \|\Delta\, \mathbf{q}\|^2 + \|\nabla\, p\|^2\right), \\ &\qquad \text{(by (3.20))} \\ &\quad = C\, \left(\nu^2\, \|\nabla \cdot \underline{\mathbf{U}}\|^2 + \|\nabla\, p\|^2\right). \end{aligned}$$

This proves (3.18) and hence the theorem for simply connected Ω.

The proof for general Ω, that is, when we assume only that $\partial\Omega$ is $C^{1,1}$, now follows by an argument similar to the proof of Theorem 3.7 in [15]. ∎

COROLLARY 3.1. *Consider the following case: $\delta = 0$, the domain Ω is a bounded convex polyhedron, and regularity bound* (2.10) *holds. Then there exists a constant C independent of ν such that for any $(\underline{\mathbf{U}}, \mathbf{u}, p) \in \mathbf{V}_3$ we have*

$$\frac{1}{C}\, \left(\nu^2\, (\|\underline{\mathbf{U}}\|_1^2 + \|d^{-1}\mathrm{tr}\underline{\mathbf{U}}\|^2) + \nu^2\, \|\mathbf{u}\|_1^2 + \|p\|_1^2\right) \leq G_3(\underline{\mathbf{U}}, \mathbf{u}, p; \mathbf{0}, 0) \tag{3.22}$$

and

$$G_2(\underline{\mathbf{U}}, \mathbf{u}, p; \mathbf{0}, 0) \leq C\, \left(\nu^2\, (\|\underline{\mathbf{U}}\|_1^2 + \|d^{-1}\mathrm{tr}\underline{\mathbf{U}}\|^2) + \nu^2\, \|\mathbf{u}\|_1^2 + \|p\|_1^2\right). \tag{3.23}$$

Proof. The proof is analogous to that of Theorem 3.2, with fairly straightforward modifications to the last string of inequalities. ∎

Remark 3.1. Theorem 3.2 establishes an equivalence relation between G_2 and the diagonal $H^1(\Omega)^{n^2+n+1}$ norm. Loosely speaking, this means that the behavior of the errors arising due to discretization and algebraic iteration can be treated *essentially* as if the variables are decoupled. This is not quite true of the equivalence established in Corollary 3.1 because U_{11} and U_{22} are strongly coupled in the trace term. However, this *intervariable* dependence can be eliminated by the simple rotation of variables:

$$\left\{ \begin{array}{rcl} U_{11}^{new} & = & (U_{11} + U_{22})/\sqrt{2}, \\ U_{22}^{new} & = & (U_{11} - U_{22})/\sqrt{2}. \end{array} \right. \tag{3.24}$$

Remark 3.2. An important characteristic of G_2 (and G_3) is its insensitivity to certain scaling of its defining terms. Specifically, consider the splitting

$$G_2(\underline{\mathbf{U}}, \mathbf{u}, p;\, \mathbf{f}, g) = G_2^{(1)}(\underline{\mathbf{U}}, p;\, \mathbf{f}, g) + G_2^{(2)}(\mathbf{u};\, \underline{\mathbf{U}}, p, g),$$

where

$$G_2^{(1)}(\underline{\mathbf{U}}, p;\, \mathbf{f}, g) = \|\mathbf{f} + \nu\,(\nabla\cdot\underline{\mathbf{U}})^t - \nabla\, p\|^2 + \nu^2\,\|\nabla\times\underline{\mathbf{U}}\|^2 + \nu^2\,\|\nabla \mathrm{tr}\,\underline{\mathbf{U}} + \delta\nabla\, p - \nabla g\|^2$$

and

$$G_2^{(2)}(\mathbf{u};\, \underline{\mathbf{U}}, p, g) = \nu^2\,\|\underline{\mathbf{U}} - \nabla\,\mathbf{u}^t\|^2 + \nu^2\,\|\nabla\cdot\mathbf{u} + \delta\, p - g\|^2.$$

It follows directly from (3.17) and (3.18) that $G_2^{(1)}(\underline{\mathbf{U}}, p;\, \mathbf{0}, 0)$ is uniformly equivalent to $\nu^2\,\|\underline{\mathbf{U}}\|_1^2 + \|p\|_1^2$, and it is immediate that $G_2^{(2)}(\mathbf{u};\, \underline{\mathbf{0}}, 0, 0)$ is uniformly equivalent to $\nu^2\,\|\mathbf{u}\|_1^2$. The practical implication is that the Stokes equations may be solved optimally and uniformly in a *two-stage* process that involves first minimizing $G_2^{(1)}(\underline{\mathbf{U}}, p;\, \mathbf{f}, g)$ over $(\underline{\mathbf{U}}, p) \in \underline{\mathbf{V}}_0 \times (H^1(\Omega)/\Re)$, then fixing $(\underline{\mathbf{U}}, p)$ and minimizing $G_2^{(2)}(\mathbf{u};\, \underline{\mathbf{U}}, p, g)$ over $\mathbf{u} \in H_0^1(\Omega)^n$. It is clear that the accuracy obtained in the first stage for $(\underline{\mathbf{U}}, p)$ is more than enough to achieve similar accuracy in the second stage for $\mathbf{u}$, and that the second stage can be avoided if velocities are not needed. These important practical advantages are a result of the more general property that the coupling between $(\underline{\mathbf{U}}, p)$ and $\mathbf{u}$ is subdominant in the sense of order of the associated differential operators (i.e., the second-order normal equations associated with the least-squares principle for G_2 have only first-order differential operators appearing in the off-diagonal blocks connecting $(\underline{\mathbf{U}}, p)$ and $\mathbf{u}$). Analogous properties hold for G_3.

Remark 3.3. The H^{-1} functional G_1 has the possible advantage over the L^2 functionals G_2 and G_3 that it can be applied when $\mathbf{f}$ in (2.1) is in $H^{-1}(\Omega)$ but not in $L^2(\Omega)$. This generality comes at the cost of involving inverses in the functional evaluation (although they can be treated effectively if not somewhat expensively in the discretization; cf. [5] and [8]). There are other, possibly more serious, practical disadvantages with G_1 that stem from the delicate balance needed between the three terms in its definition (3.1). For example, it is not possible to use G_1 in a two-stage process, as we described for G_2, because all variables are strongly coupled through each of the defining terms. Unfortunately, the character of G_1 depends too critically on proper scaling of these terms. However, an H^{-1} functional that does not suffer such scaling limitations is obtained by appealing to the extended system (2.8):

$$\begin{aligned} G_4(\underline{\mathbf{U}},\, \mathbf{u},\, p;\, \mathbf{f},\, g) = {} & \|\mathbf{f} + \nu\,(\nabla\cdot\underline{\mathbf{U}})^t - \nabla\, p\|_{-1}^2 + \nu^2\,\|\underline{\mathbf{U}} - \nabla\,\mathbf{u}^t\|^2 \\ & + \nu^2\,\|\nabla\cdot\mathbf{u} + \delta\, p - g\|^2 + \nu^2\,\|\nabla\times\underline{\mathbf{U}}\|_{-1}^2 + \nu^2\,\|\mathrm{tr}\,\underline{\mathbf{U}} + \delta\, p - g\|^2. \end{aligned} \tag{3.25}$$

Here, the H^{-1} norm on the div term in G_4 corresponds to the dual of $H_0^1(\Omega)^n$, as before. But for the curl term, it is probably best to interpret the H^{-1} norm as that corresponding to the dual of $\mathcal{V}_0$. In any case, Theorem 3.1 and the boundedness of the curl operator as a mapping from $L^2(\Omega)^n$ to $H^{-1}(\Omega)^n$ and the trace operator from $L^2(\Omega)^{n^2}$ to $L^2(\Omega)$ imply that $G_4(\underline{\mathbf{U}},\, \mathbf{u},\, p;\, \mathbf{0},\, 0)$ is uniformly equivalent to $\nu^2\|\underline{\mathbf{U}}\|^2 + \nu^2\|\mathbf{u}\|_1^2 + \|p\|^2$ on $\mathbf{V}_1$. As with G_1, H^2 regularity assumption (2.9) is not needed for this equivalence to hold. Now, the required number of inverse evaluations for G_4 is double that for G_1, but this cost is outweighed by its superior scale properties

analogous to those of G_2 suggested in the previous remark. G_4 is especially attractive because basic iterative methods like steepest descent or conjugate gradients can be used for efficient approximation of $\underline{\mathbf{U}}$ and p in a corresponding two-stage process. (Wellposedness of the first-stage functional

$$G_4^{(1)}(\underline{\mathbf{U}}, p; \mathbf{f}, g) = \|\mathbf{f} + \nu\,(\nabla\cdot\underline{\mathbf{U}})^t - \nabla\,p\|_{-1}^2 + \nu^2\,\|\nabla\times\underline{\mathbf{U}}\|_{-1}^2 + \nu^2\,\|\mathrm{tr}\,\underline{\mathbf{U}} + \delta\,p - g\|^2$$

requires proper boundary conditions on $\underline{\mathbf{U}}$, which is easily done by restricting $G_4^{(1)}$ to $H_0(\mathbf{curl};\,\Omega)^n \times L_0^2(\Omega)$.)

Remark 3.4. An attractive feature of least-squares methods is the freedom to incorporate equations and boundary conditions in the functional or to impose them on the space. For example, we can impose the trace equation in (2.7) by restricting the space to the subset of variables that satisfy this equation. For G_2, this amounts to minimizing

$$\begin{aligned} G_2(\underline{\mathbf{U}},\,\mathbf{u},\,p;\,\mathbf{f},\,g) = {} & \|\mathbf{f} + \nu\,(\nabla\cdot\underline{\mathbf{U}})^t - \nabla\,p\|^2 + \nu^2\,\|\underline{\mathbf{U}} - \nabla\,\mathbf{u}^t\|^2 \\ & + \nu^2\,\|\nabla\cdot\mathbf{u} + \delta\,p - g\|^2 + \nu^2\,\|\nabla\times\underline{\mathbf{U}}\|^2 \end{aligned} \tag{3.26}$$

over $\mathbf{V}_2^g = \{(\underline{\mathbf{U}},\,\mathbf{u},\,p) \in \mathbf{V}_2 \,:\, \mathrm{tr}\underline{\mathbf{U}} + \delta p = g\}$. Note that the norm equivalence asserted in Theorem 3.2 still holds in this case simply because $\mathbf{V}_2^g$ is a subset of $\mathbf{V}_2$. This is true even for the case that $\delta = 0$ and Ω is a bounded convex polyhedron, provided $d^{-1}g \in L^2(\Omega)$, because then $\mathbf{V}_2^g$ is a subset of $\mathbf{V}_3$ on which G_2 and G_3 agree. Note that Remarks 3.1 and 3.2 also apply here, again simply because $\mathbf{V}_2^g$ is a subset of $\mathbf{V}_2$. Possible advantages of this approach include that $\mathbf{V}_2^g$ can be represented by fewer dependent variables than can $\mathbf{V}_2$ (when $\delta > 0$, variable $p = d^{-1}(g - \mathrm{tr}\underline{\mathbf{U}})$ can be eliminated; when $\delta = 0$, we can use the rotation defined in (3.24), which allows elimination of $U_{11}^{new} = g$); the term in G_3 involving d is eliminated; and the velocity flux variables satisfy the continuity equation exactly (which might be an important consideration in some applications).

Remark 3.5. We now show that the last two terms in the definition of G_2 are necessary for the bound (3.14) to hold, even with the extra boundary condition $\mathbf{n} \times \underline{\mathbf{U}} = \underline{\mathbf{0}}$. We consider the Stokes equations, so that $\delta = 0$. Suppose first that we omit the term $\|\nabla\times\underline{\mathbf{U}}\|^2$ but include the term $\|\nabla\mathrm{tr}\,\underline{\mathbf{U}}\|^2$. We offer a two-dimensional counterexample; a three-dimensional counterexample can be constructed in a similar manner. Let $\nu = 1$, $\mathbf{u} = \mathbf{0}$, and $p = 0$. Choose any $\omega \in \mathcal{D}(\Omega)$ such that $\Delta\nabla\omega \neq \mathbf{0}$ and define

$$\underline{\mathbf{U}} \equiv \nabla^{\perp}(\nabla\omega)^t.$$

Clearly, $\mathbf{n} \times \underline{\mathbf{U}} = \underline{\mathbf{0}}$. It is easy to show that

$$\nabla\cdot\underline{\mathbf{U}} = \mathbf{0} \quad \text{and} \quad \mathrm{tr}\,\underline{\mathbf{U}} = \nabla\times(\nabla\omega) = 0.$$

However,

$$(\nabla\times\underline{\mathbf{U}})^t = \Delta\nabla\omega \neq \mathbf{0}$$

by construction. Thus,

$$G_2(\underline{\mathbf{U}},\,\mathbf{u},\,p;\,\mathbf{0}) = \|\underline{\mathbf{U}}\|^2,$$

which cannot bound $\|\underline{\mathbf{U}}\|_1^2$. That is, since $\omega \in \mathcal{D}(\Omega)$ is arbitrary, we may choose it so oscillatory that $\|\underline{\mathbf{U}}\|_1/\|\underline{\mathbf{U}}\|$ is as large as we like. This prevents the bound (3.14) from

holding. Next, suppose that we include the $\|\nabla\times \underline{\mathbf{U}}\|^2$ term but omit the $\|\nabla \mathrm{tr}\, \underline{\mathbf{U}}\|^2$ term. Now set $\Omega = (0,\, 1)^2$, $\nu = 1$, $\mathbf{u} = \mathbf{0}$, and $p = \cos(k\pi x_1)\sin(\pi x_2)$, and choose q_i to satisfy

$$\left\{ \begin{array}{rcll} -\Delta\, q_i & = & -\partial_i p & \text{in} \quad \Omega, \\ q_i & = & 0 & \text{on} \quad \partial\Omega, \end{array} \right.$$

for $i = 1,\, 2$. Then

$$q_1 = \frac{k}{\pi(k^2+1)} \sin(k\pi x_1)\sin(\pi x_2).$$

We also know that

$$\|\nabla\, q_2\| \le C\, \|\partial_2 p\| = \|\pi \cos(k\pi x_1)\cos(\pi x_2)\| \le C,$$

where C is independent of k. Now set

$$\mathbf{U}_i = \nabla\, q_i$$

for $i = 1,\, 2$. Then $\mathbf{n}\times\mathbf{U}_i = \mathbf{0}$ and

$$G_2(\underline{\mathbf{U}},\, \mathbf{u},\, p;\, \mathbf{0}) = \|\Delta\, \mathbf{q} - \nabla\, p\|^2 + \|\nabla\, \mathbf{q}\|^2 = \|\nabla\, \mathbf{q}\|^2 \le C,$$

where C is independent of k. On the other hand, we have

$$\|p\|_1 \ge C\, k,$$

which again prevents the bound (3.14) from holding.

4. Concluding remarks. We needed full regularity assumption (2.9) (and (2.10)) only in Theorem 3.2 (and Corollary 3.1) to obtain full H^1 product ellipticity of functional G_2 in (3.2) (and G_3 in (3.3)). This somewhat restrictive assumption is not necessary either for functional G_1 in (3.1) or for functional G_4 in (3.25), both of which support efficient practical algorithms (the H^{-1} norms can be replaced by discrete H^{-1} norms or simpler mesh weighted norms; see [5] and [8] for analogous H^{-1} norm algorithms) and which have the weaker norm equivalence assured by Theorem 3.1 and Remark 3.3.

Nevertheless, the principal result of this paper is Theorem 3.2 (and Corollary 3.1), which establishes full H^1 product ellipticity of least-squares functional G_2 (and G_3) for the generalized Stokes system. Since we have assumed full H^2 regularity of the original Stokes (linear elasticity) equations, we may then use this result to establish optimal finite element approximation estimates and optimal multiplicative and additive multigrid convergence rates. This can be done in precisely the same way that these results were established for general second-order elliptic equations (see [11, sections 3–5]). We therefore omit this development here. However, it is important to recognize that the ellipticity property is independent of the Reynolds parameter ν (Poisson ratio $\frac{\lambda}{2(\lambda+\mu)}$). This automatically implies that the optimal finite element discretization error estimates and multigrid convergence factor bounds are uniform in ν ($\frac{\lambda}{2(\lambda+\mu)}$). At first glance, it might appear that the scaling of some of the H^1 product norm components might create a scale dependence of our discretization and algebraic convergence estimates. However, the results in [11] are based only on assumptions posed in an unscaled H^1 product norm, in which the individual variables

are completely decoupled; and since the constant ν appears only as a simple factor in individual terms of the scaled H^1 norm, these assumptions are equally valid in this case. On the other hand, for problems where the necessary H^1 scaling is not (essentially) constant, extension of the theory of sections 3–5 of [11] is not straightforward. Such is the case for convection-diffusion equations, which will be treated in a forthcoming paper.

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