A Finite Element Method Using Singular Functions for Poisson Equations: Mixed Boundary Conditions

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Abstract

In [7], we proposed a new finite element method to compute singular solutions of Poisson equations on a polygonal domain with re-entrant angles. Singularities are eliminated and only the regular part of the solution that is in $H^2$ is computed. The stress intensity factor and the solution can be computed as a post processing step. This method is extended to problems with crack singularities and to a higher-order method for smooth data in [9]. In this paper, we study the Poisson equation with mixed boundary conditions. Examples with various singular points and numerical results are presented.

1 Introduction

Assume that $\Omega \subset \mathbb{R}^2$ is a polygonal domain. Let $\Gamma_D$ and $\Gamma_N$ be a partition of the boundary of $\Omega$ such that $\partial \Omega = \Gamma_D \cup \Gamma_N$ and $\Gamma_D \cap \Gamma_N = \emptyset$. For simplicity, assume that $\Gamma_D$ is not empty (i.e., $\text{meas}(\Gamma_D) \neq 0$). Let $\nu$ denote the outward unit vector normal to the boundary. For a given function $f \in L^2(\Omega)$, consider the Poisson equation with homogeneous mixed boundary conditions:

$$
\begin{align*}
-\Delta u &= f \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \Gamma_D, \\
\frac{\partial u}{\partial \nu} &= 0 \quad \text{on } \Gamma_N,
\end{align*}
$$

(1.1)

where $\Delta$ stands for the Laplacian operator. Solution of (1.1) has singular behavior near corners even when $f$ is very smooth. Such singular behavior affects the accuracy of the finite element method throughout the whole domain.

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Solutions of many elliptic boundary value problems on polygonal domains have a singular function representation: the sum of singular functions and the regular part of the solution. This property has been explored in several ways to design accurate finite element methods in the presence of corner singularities. One approach is the so-called Singular Function Method (SFM) (see, e.g., [13]) that augments singular functions to both the trial and test spaces. Another is the so-called Dual Singular Function Method (DSFM) (see, e.g., [11, 4, 3, 6, 12]) that augments singular functions to the trial space and the corresponding dual singular functions to the test space.

Recently we also use this property in order to calculate accurate finite element approximations to both the solution and the stress intensity factors, that are coefficients of singular functions in the singular function representation of the solution. By using the dual singular functions and a particularly chosen cut-off function, we are able to deduce a well-posed variational problem for the regular part of the solution. Hence, standard finite element approximation of this problem is optimally accurate. The stress intensity factors and the solution can then be computed with optimal accuracy as a post processing step. The Poisson equation with homogeneous Dirichlet boundary conditions was studied in [7]. This note studies mixed boundary conditions. Singular functions for mixed boundary conditions differ from those for Dirichlet boundary conditions in both form and angles of corners.

In section 2, singular function representation of the solution for various boundary conditions is presented and a variational problem for regular part of the solution is derived. In section 3, we introduce a finite element approximation and estimate its error bound. Finally, in section 4, we present two examples with various singular functions and their numerical results.

We will use the standard notation and definitions for the Sobolev spaces $H^t(B)$ for $t \geq 0$; the standard associated inner products are denoted by $(\cdot, \cdot)_{t,B}$, and their respective norms and seminorms are denoted by $\| \cdot \|_{t,B}$ and $| \cdot |_{t,B}$. The space $L^2(B)$ is interpreted as $H^0(B)$, in which case the inner product and norm will be denoted by $(\cdot, \cdot)_B$ and $\| \cdot \|_B$, respectively. $H^1_D(B) = \{ u \in H^1(B) : u = 0 \text{ on } \Gamma_D \}$.

## 2 Singular Function Representations

Let $\omega_1, \cdots, \omega_M$ be internal angles of $\Omega$ satisfying

\[
\begin{align*}
\frac{\pi}{2} < \omega_j < 2\pi & \quad \text{if boundary condition changes its type}, \\
\pi < \omega_j < 2\pi & \quad \text{otherwise}
\end{align*}
\]

and denote by $v_j$ ($j = 1, \cdots, M$) the corresponding vertices. Let the polar co-ordinates $(r_j, \theta_j)$ be chosen at the vertex $v_j$ so that the internal angle $\omega_j$ is spanned counterclockwise by two half-lines $\theta_j = 0$ and $\theta_j = \omega_j$. Below is a list of singular functions at $v_j$ depending on boundary conditions:
• **D/D** If \( \omega_j > \pi \), there is a singular function of the form

\[
s_{j,1}(r_j, \theta_j) = r_j^{\frac{\pi}{\omega_j}} \sin \frac{\pi \theta_j}{\omega_j};
\]  
(2.1)

• **N/N** If \( \omega_j > \pi \), there is a singular function of the form

\[
s_{j,1}(r_j, \theta_j) = r_j^{\frac{\pi}{\omega_j}} \cos \frac{\pi \theta_j}{\omega_j};
\]  
(2.2)

• **D/N** If \( \frac{\pi}{2} < \omega_j \leq \frac{3\pi}{2} \), there is a singular function of the form

\[
s_{j,\frac{1}{2}}(r_j, \theta_j) = r_j^{\frac{\pi}{2\omega_j}} \sin \frac{\pi \theta_j}{2\omega_j};
\]  
(2.3)

If \( \frac{3\pi}{2} < \omega_j < 2\pi \), there are two singular functions of the form

\[
s_{j,\frac{1}{2}}(r_j, \theta_j) = r_j^{\frac{\pi}{2\omega_j}} \sin \frac{\pi \theta_j}{2\omega_j} \quad \text{and} \quad s_{j,\frac{3}{2}}(r_j, \theta_j) = r_j^{\frac{3\pi}{2\omega_j}} \sin \frac{3\pi \theta_j}{2\omega_j};
\]  
(2.4)

• **N/D** If \( \frac{\pi}{2} < \omega_j \leq \frac{3\pi}{2} \), there is a singular function of the form

\[
s_{j,\frac{1}{2}}(r_j, \theta_j) = r_j^{\frac{\pi}{2\omega_j}} \cos \frac{\pi \theta_j}{2\omega_j};
\]  
(2.5)

If \( \frac{3\pi}{2} < \omega_j < 2\pi \), there are two singular functions of the form

\[
s_{j,\frac{1}{2}}(r_j, \theta_j) = r_j^{\frac{\pi}{2\omega_j}} \cos \frac{\pi \theta_j}{2\omega_j} \quad \text{and} \quad s_{j,\frac{3}{2}}(r_j, \theta_j) = r_j^{\frac{3\pi}{2\omega_j}} \cos \frac{3\pi \theta_j}{2\omega_j}.
\]  
(2.6)

Here, D/D and N/N mean that type of boundary conditions remains unchanged while D/N and N/D mean that type of boundary conditions changes passing the vertex \( v_j \). For convenience, we denote index set of singular functions by \( L_j \). Hence,

\[
L_j = \begin{cases} 
\{1\} & \text{for (2.1) and (2.2)}, \\
\{\frac{1}{2}\} & \text{for (2.3) and (2.5)}, \\
\{\frac{1}{2}, \frac{3}{2}\} & \text{for (2.4) and (2.6)}.
\end{cases}
\]

It is easy to see that \( s_{j,\frac{3}{2}} \in H^{1+\frac{3\pi}{2\omega_j} - \varepsilon}(\Omega) \), \( s_{j,1} \in H^{1+\frac{\pi}{\omega_j} - \varepsilon}(\Omega) \), and \( s_{j,\frac{1}{2}} \in H^{1+\frac{\pi}{2\omega_j} - \varepsilon}(\Omega) \) for any \( \varepsilon > 0 \). Hence, singular functions at the vertex \( v_j \) belong to either \( H^{1+\frac{\pi}{2\omega_j} - \varepsilon}(\Omega) \)
for D/D and N/N vertex or $H^{1+\frac{\pi}{\omega_j} - \varepsilon}(\Omega)$ for D/N and N/D vertex. This indicates that the solution of Poisson equation (1.1) is in $H^{1+\frac{\pi}{\omega_j} - \varepsilon}$ where $\omega = \max_{1 \leq j \leq M} \hat{\omega}_j$ and

$$
\hat{\omega}_j = \begin{cases} 
\omega_j & \text{if } v_j \text{ is D/D or N/N vertex,} \\
2\omega_j & \text{if } v_j \text{ is D/N or N/D vertex.}
\end{cases}
$$

To deduce an equation for the regular part of the solution, we need to use the so-called dual singular functions that are defined as follows: for $l \in L_j$,

$$
s_{j,-l}(r_j, \theta_j) = \frac{-l\pi}{\omega_j} \sin \frac{l\pi}{\omega_j} \theta_j \quad \text{and} \quad s_{j,-l}(r_j, \theta_j) = \frac{-l\pi}{\omega_j} \cos \frac{l\pi}{\omega_j} \theta_j,
$$

are the dual singular functions corresponding to

$$
s_{j,l}(r_j, \theta_j) = \frac{l\pi}{\omega_j} \sin \frac{l\pi}{\omega_j} \theta_j \quad \text{and} \quad s_{j,l}(r_j, \theta_j) = \frac{l\pi}{\omega_j} \cos \frac{l\pi}{\omega_j} \theta_j
$$

respectively. We will also need cut-off functions. To this end, set

$$
B_j(t_1; t_2) = \{ (r_j, \theta_j) : t_1 < r_j < t_2 \text{ and } 0 < \theta_j < \omega_j \} \cap \Omega \quad \text{and} \quad B_j(t_1) = B_j(0; t_1).
$$

A family of cut-off functions of $r_j, \eta_{\rho_j}(r_j)$, is then defined as follows:

$$
\eta_{\rho_j}(r_j) = \begin{cases} 
1 & \text{in } B_j(\frac{1}{2}\rho_j R), \\
\frac{15}{16} \left\{ \frac{8}{15} - \left( \frac{4r_j}{\rho_j R} - 3 \right) + \frac{2}{3} \left( \frac{4r_j}{\rho_j R} - 3 \right)^3 - \frac{1}{5} \left( \frac{4r_j}{\rho_j R} - 3 \right)^5 \right\} & \text{in } \bar{B}_j(\frac{1}{2}\rho_j R; \rho_j R), \\
0 & \text{in } \Omega \setminus \bar{B}_j(\rho_j R)
\end{cases}
$$

where $\rho_j$ is a parameter in $(0, 2]$ and $R \in \mathcal{R}$ is a fixed number so that the $\eta_{\rho_j} s_{j,l}$ has the same boundary condition as $u$. We assume that $R$ is small enough so that the intersection of either $B_j(\rho_j R)$ and $B_i(2R)$ or $B_j(2R)$ and $B_i(\rho_j R)$ for $j \neq i$ is empty.

It is well known ([1, 10, 11]) that the solution of problem (1.1) has the following singular function representation:

$$
u = w + \sum_{j=1}^{M} \sum_{l \in L_j} \lambda_{j,l} \eta_{\rho_j}(r_j) s_{j,l}(r_j, \theta_j)
$$

where $w \in H^2(\Omega) \cap H^1_D(\Omega)$ is the regular part of the solution and $\lambda_{j,l} \in \mathcal{R}$ are the stress intensity factors that can be expressed in terms of $u$ by the following extraction formulas ([5, 14]):

$$
\lambda_{j,l} = \frac{1}{l\pi} \left( \int_{\Omega} f \eta_{\rho_j} s_{j,-l} \, dx + \int_{\Omega} u \Delta(\eta_{\rho_j} s_{j,-l}) \, dx \right).
$$
Moreover, the following regularity estimate holds:

\[ \|w\|_2 + \sum_{j=1}^{M} \sum_{l \in L_j} |\lambda_{j,l}| \leq C_R \|f\|. \quad (2.11) \]

In the remainder of this section, we derive a well-posed problem for \( w \). To this end, assume that \( \rho_j \) in (2.9) belongs to \((0, 1]\) and denote cut-off functions with bigger supports by

\[ \eta^*(r_j) = \eta_2(r_j). \]

Choosing \( \eta_\rho(r_j) = \eta^*(r_j) \) in (2.10) gives

\[ \lambda_{j,l} = \frac{1}{l\pi} (u, \Delta(\eta^* s_{j,-l})) + \frac{1}{l\pi} (f, \eta^* s_{j,-l}). \]

Substituting \( u = w + \sum_{i=1}^{M} \sum_{k \in L_i} \lambda_{i,k} \eta_{\rho_i}(r_i) s_{i,k}(r_i, \theta_i) \) into the above equation yields

\[ \lambda_{j,l} = \frac{1}{l\pi} (w, \Delta(\eta^* s_{j,-l})) + \frac{1}{l\pi} (f, \eta^* s_{j,-l}) + \frac{1}{l\pi} \sum_{i=1}^{M} \sum_{k \in L_i} \lambda_{i,k} (\eta_{\rho_i} s_{i,k}, \Delta(\eta^* s_{j,-l})). \quad (2.12) \]

When \( i = j \), the support of \( \eta_{\rho_i}(r_j) \) for \( 0 < \rho_j \leq 1 \) is \( B_j(\rho_j R) \) on which \( \eta^* = 1 \). Since \( s_{j,-l} \) is harmonic, then for all \( k \in L_j \),

\[ (\eta_{\rho_i} s_{i,k}, \Delta(\eta^* s_{j,-l})) = 0. \]

When \( i \neq j \), by the assumption that \( B_i(\rho R) \cap B_j(2R) = \emptyset \) we have that

\[ (\eta_{\rho_i} s_{i,k}, \Delta(\eta^* s_{j,-l})) = 0, \quad \forall \ k \in L_i. \]

Hence, we have established the following extraction formulas of \( \lambda_{j,l} \) in terms of \( w \):

\[ \lambda_{j,l} = \frac{1}{l\pi} (w, \Delta(\eta^* s_{j,-l}))_{B_j(R;2R)} + \frac{1}{l\pi} (f, \eta^* s_{j,-l})_{B_j(2R)}. \quad (2.13) \]

Using (2.13) and substituting (2.9) into the Poisson equation, we obtain an integro-differential equation for \( w \):

\[ -\Delta w - \sum_{j=1}^{M} \sum_{l \in L_j} \frac{1}{l\pi} (w, \Delta(\eta^* s_{j,-l}))_{B_j(R;2R)} \Delta(\eta_{\rho_j} s_{j,l}) = f + \sum_{j=1}^{M} \sum_{l \in L_j} \frac{1}{l\pi} (f, \eta^* s_{j,-l})_{B_j(2R)} \Delta(\eta_{\rho_j} s_{j,l}) \quad \text{in} \ \Omega. \]

Multiplying the above equation by a test function \( v \in H^1_D(\Omega) = \{ v \in H^1(\Omega) : v = 0 \text{ on } \Gamma_D \} \), integrating over \( \Omega \), and using integration by parts lead to the following variational problem: finding \( w \in H^2(\Omega) \cap H^1_D(\Omega) \) such that

\[ a(w, v) = g(v) \quad \forall v \in H^1_D(\Omega), \quad (2.14) \]
where the bilinear form \(a(\cdot, \cdot)\) and linear form \(g(\cdot)\) are defined by

\[
a(w, v) = a^s(w, v) + b(w, v), \quad a^s(w, v) = (\nabla w, \nabla v),
\]

\[
b(w, v) = -\sum_{j=1}^{M} \sum_{l \in L_j} \frac{1}{l\pi} (w, \Delta(\eta^s s_j, -l))_{B_j(R;2R)}(\Delta(\eta^s s_j, l), v)_{B_j(\rho_j R; \rho_j R)}
\]

and

\[
g(v) = (f, v) + \sum_{j=1}^{M} \sum_{l \in L_j} \frac{1}{l\pi} (f, \eta^s s_j, -l)_{B_j(2R)}(\Delta(\eta^s s_j, l), v)_{B_j(\rho_j R; \rho_j R)}
\]

Note that the second terms in the respective bilinear and linear forms provide a singular correction so that \(w \in H^2(\Omega)\) for \(f \in L^2(\Omega)\). Note also that the bilinear forms \(a(\cdot, \cdot)\) are not symmetric.

**Lemma 2.1** For any \(0 < \rho \leq 1\), we have that

\[
\|\Delta(\eta^s s_j, -l)\|_{B_j(R;2R)} \leq C_1 R^{\frac{j}{2} - 1}
\]

with \(C_1 = \sqrt{\frac{120\omega_j}{\pi}}\) and that

\[
\|\eta^s s_j, l\|_{B_j(\rho_j R)} \leq C_2(\rho_j R)^{\frac{j}{2} + \frac{j}{2}} \quad \text{and} \quad \|\nabla(\eta^s s_j, l)\|_{B_j(\rho_j R)} \leq C_3(\rho_j R)^{\frac{j}{2} - \frac{j}{2}}
\]

with \(C_2 = \frac{\omega_j}{2\sqrt{r_j + \omega_j}}\) and \(C_3 = (\frac{C^2\omega_j^2}{4 |r_j + \omega_j|}) (1 - 2^{-2(\frac{j}{2} + 1)}) + \left(\frac{j}{2}\right)^2\).

**Proof.** This lemma can be established by an elementary calculation.

In a similar fashion as in [7], we can prove the coercivity and continuity of the bilinear form \(a(\cdot, \cdot)\) and the well posed-ness of problem (2.14).

**Lemma 2.2** For \(0 < \rho \leq 1\), the bilinear forms \(a(\cdot, \cdot)\) are continuous and coercive in \(H_D^1(\Omega)\); i.e. there exist positive constants \(\alpha, K_1,\) and \(K_2\) such that

\[
\alpha\|\phi\|^2_1 \leq a(\phi, \phi) + K_1\|\phi\|^2
\]

for all \(\phi \in H_D^1(\Omega)\) and that

\[
a(\phi, \psi) \leq K_2\|\phi\|_1\|\psi\|_1
\]

for all \(\phi\) and \(\psi\) in \(H_D^1(\Omega)\).

**Theorem 2.1** For \(0 < \rho \leq 1\), we have that

(1) if \(f \in L^2(\Omega)\), then problem (2.14) has a unique solution \(w \in H^2(\Omega) \cap H_D^1(\Omega)\).

(2) there exists a positive constant \(\gamma\) such that

\[
\gamma\|\phi\|_1 \leq \sup_{\psi \in H^1_D(\Omega)} \frac{a(\phi, \psi)}{\|\psi\|_1}
\]

for any \(\phi \in H_D^1(\Omega)\).
3 Finite Element Approximation

This section presents standard finite element approximation on a quasi-uniform grid for $w$ based on the variational problem in (2.14). Approximations to the stress intensity factors and the solution of problem (1.1) can then be calculated according to (2.13) and (2.9), respectively. Error estimates are established in Theorem 3.1.

Let $T_h$ be a partition of the domain $\Omega$ into triangular finite elements; i.e., $\Omega = \bigcup_{K \in T_h} K$ with $h = \max\{\text{diam } K : K \in T_h\}$. Assume that the triangulation $T_h$ is regular. Denote continuous piecewise linear finite element space by $V_h = \{ \phi_h \in C^0(\Omega) : \phi_h|_K \text{ is linear } \forall K \in T_h \text{ and } \phi_h = 0 \text{ on } \Gamma_D \} \subset H^1_D(\Omega)$.

It is well known that
\[
\inf_{\phi_h \in V_h} (\| \phi - \phi_h \| + h|\phi - \phi_h|_1) \leq C_A h^{1+t}\|\phi\|_{1+t,\Omega} \tag{3.1}
\]
for any $\phi \in H^1_D(\Omega) \cap H^{1+t}(\Omega)$ and $0 \leq t \leq 1$. The finite element approximation to problem (2.14) is to find $w_h \in V_h$ such that
\[
a(w_h, v) = g(v) \quad \forall v \in V_h. \tag{3.2}
\]

Approximations to the $\lambda_{j,l}^h$ and the solution are calculated as follows:
\[
\lambda_{j,l}^h = \frac{1}{l \pi} (w_h, \Delta(\eta^* s_{j,-l}))_{B_j(2R)} + \frac{1}{l \pi} (f, \eta^* s_{j,-l})_{B_j(2R)} \tag{3.3}
\]
and
\[
u_h = w_h + \sum_{j=1}^M \sum_{l \in L_j} \lambda_{j,l}^h \eta_{p_j}(r_j) s_{j,l}(r_j, \theta_j). \tag{3.4}
\]

In order to establish the error bound in the $L^2$ norm, we consider the following adjoint problem of (2.14) with a simplified linear form: find $z \in H^1_D(\Omega)$ such that
\[
a(v, z) = (w - w_h, v) \quad \forall v \in H^1_D(\Omega). \tag{3.5}
\]
The next lemma establishes the well-posedness of problem (3.5) and provides the regularity estimate for $z$.

**Lemma 3.1** For $0 < \rho_j \leq 1$, problem (3.5) has a unique solution $z$ in $H^1_D(\Omega)$. Moreover, there is a singular function representation
\[
z = w_z + \sum_{j=1}^M \sum_{l \in L_j} \lambda_{j,l}^z \eta_{p_j} s_{j,l} \tag{3.6}
\]
where $w_z \in H^2(\Omega) \cap H^1_D(\Omega)$ and $\lambda_{j,l}^z \in R$ satisfy the regularity estimate
\[
\|w_z\|_2 + \sum_{j=1}^M \sum_{l \in L_j} |\lambda_{j,l}^z| \leq C'_R \|w - w_h\|. \tag{3.7}
\]
Proof. Similar to Theorem 2.1, the adjoint problem in (3.5) has a unique solution in \( H^1_D(\Omega) \) and there exists a positive constant \( \gamma' \) such that

\[
\gamma' \| \psi \|_1 \leq \sup_{\phi \in H^1_D(\Omega)} \frac{a(\phi, \psi)}{\| \phi \|_1} \quad \forall \psi \in H^1_D(\Omega).
\]

Let \( z \) be the solution of (3.5), by the Cauchy-Schwarz inequality we then have that

\[
\| z \|_1 \leq \frac{1}{\gamma} \sup_{\phi \in H^1_b(\Omega)} \frac{a(\phi, z)}{\| \phi \|_1} \leq \frac{1}{\gamma} \sup_{\phi \in H^1_b(\Omega)} \frac{(w - w_h, \phi)}{\| \phi \|_1} \leq \frac{1}{\gamma} \| w - w_h \|. \tag{3.8}
\]

It is easy to check that the solution, \( z \in H^1_D(\Omega) \), of problem (3.5) satisfies

\[
\Delta z = \sum_{j=1}^M \sum_{l \in L_j} \frac{1}{l \pi} (\nabla z, \nabla (\eta_{\rho_j s_j, l})) \Delta (\eta^* s_j, -l) \quad \text{in } \Omega. \tag{3.9}
\]

Since the right-hand side of the above equation is at least in \( L^2(\Omega) \), so is \( \Delta z \). Therefore, \( z \) has the singular function representation

\[
z = w_z + \sum_{j=1}^M \sum_{l \in L_j} \lambda_{j,l}^z \eta_{\rho_j s_j, l},
\]

where \( w_z \in H^2(\Omega) \cap H^1_D \) and

\[
\| w_z \|_2 + \sum_{j=1}^M \sum_{l \in L_j} |\lambda_{j,l}^z| \leq C_R \| \Delta z \|.
\]

Now, the regularity bound in (3.7) follows from the triangle and Cauchy-Schwarz inequalities, (3.8), and Lemma 2.1 that

\[
\| w_z \|_2 + \sum_{j=1}^M \sum_{l \in L_j} |\lambda_{j,l}^z| \leq C_R \| \Delta z \|
\leq C_R \left( \sum_{j=1}^M \sum_{l \in L_j} \frac{1}{l \pi} (\nabla (\eta_{\rho_j s_j, l}), \nabla z)_{B(\rho_j R)} \| \Delta (\eta^* s_j, -l) \|_{B(R,2R)} + \| w - w_h \| \right)
\leq C_R \left( \sum_{j=1}^M \sum_{l \in L_j} \frac{C_1 C_3}{\gamma \pi} \rho_j^2 + 1 \| w - w_h \| \right).
\]

This proves the inequality in (3.7) with

\[
C'_R = C_R \left( \sum_{j=1}^M \sum_{l \in L_j} \frac{C_1 C_3}{\gamma \pi} \rho_j^2 + 1 \right)
\]

and, hence, the lemma.

Now we are ready to establish error bounds for the finite element approximations.
Theorem 3.1  
(i) For $0 < \rho_j \leq 1$, there exists a positive constant $h_0$ such that for all $h \leq h_0$ (3.2) has a unique solution $w_h$ in $V_h$. Moreover, let $w \in H^2(\Omega) \cap H^1_D(\Omega)$ be the solution of (2.14), then we have the following error estimates:

$$
\| w - w_h \|_1 \leq C_4 h \| f \| \quad \text{and} \quad \| w - w_h \| \leq C_5 h^{1+\frac{\pi}{2}} \| f \|.
$$

(3.10)

(ii) Let $\lambda_{j,t}$ and $\lambda_{j,t}^h$ be defined in (2.13) and (3.3), respectively. Then

$$
| \lambda_{j,t} - \lambda_{j,t}^h | \leq C \frac{R}{\pi} \frac{1}{\rho_j} - 1 \| w - w_h \| \leq C_6 \frac{R}{\pi} \frac{1}{\rho_j} - 1 h^{1+\frac{\pi}{2}} \| f \|.
$$

(3.11)

(iii) Let $u$ be the solution of (1.1) and $u_h$ be its approximation defined in (3.4), then we have the following error estimates:

$$
\| u - u_h \|_1 \leq C_7 h \| f \| \quad \text{and} \quad \| u - u_h \| \leq C_8 h^{1+\frac{\pi}{2}} \| f \|.
$$

(3.12)

Proof. (i) We first establish error bounds in (3.10) for any solution to problem (3.2) that may exist. Then, for $f \equiv 0$, the uniqueness of the solution to problem (2.14) and the error bound in (3.10) imply that $w_h \equiv 0$. Hence, (3.2) has a unique solution $w_h$ in $V_h$ since it is a finite dimensional problem with the same number of unknowns and equations.

To establish error bounds, note first the orthogonality property

$$
a(w - w_h, v) = 0 \quad \forall \ v \in V_h.
$$

(3.13)

By choosing $v = w - w_h$ in (3.5) and using the orthogonality property in (3.13) and the continuity bound in (2.20), we have that

$$
\| w - w_h \|^2 = a(w - w_h, z) = a(w - w_h, z - I_h z) \leq K_2 \| w - w_h \|_1 \| z - I_h z \|_1,
$$

(3.14)

where $I_h z \in V_h$ is the nodal interpolant of $z$. From the triangle inequality, approximation property (3.1), the fact that (see [2])

$$
\| \eta_{\rho_j s_j, t} - I_h (\eta_{\rho_j s_j, t}) \|_1 \leq C h^{\frac{\rho_j}{2}},
$$

and Lemma 3.1, one has

$$
\| z - I_h z \|_1 \leq \| w_z - I_h w_z \|_1 + \sum_{j=1}^M \sum_{l \in L_j} | \lambda_{j,t}^z | \| \eta_{\rho_j s_j, t} - I_h (\eta_{\rho_j s_j, t}) \|_1
$$

$$
\leq C h \| w_z \|_2 + \sum_{j=1}^M \sum_{l \in L_j} C h^{\frac{\rho_j}{2}} | \lambda_{j,t}^z | \leq C_D h^\frac{\pi}{2} \| w - w_h \|.
$$

Substituting this into (3.14) and dividing $\| w - w_h \|$ on both sides give

$$
\| w - w_h \| \leq K_2 C_D h^\frac{\pi}{2} \| w - w_h \|_1.
$$

(3.15)
Now, it follows Lemma 2.2, orthogonality property (3.13), and inequality (3.15) that for any \( v \in V_h \)

\[
\alpha \| w - w_h \|_1^2 \leq a(w - w_h, w - w_h) + K_1 \| w - w_h \|_1^2
\]

\[
= a(w - w_h, w - v) + K_1 \| w - w_h \|_1^2
\]

\[
\leq K_2 \| w - w_h \|_1 \| w - v \|_1 + K_1 (K_2 C_D h^\frac{p}{2})^2 \| w - w_h \|_1^2,
\]

which, together with approximation property (3.1), implies the validity of the first error bound in (3.10) with \( C_4 = 2\alpha^{-1} K_2 C_A C_R \) for all \( h \leq h_0 \). Here,

\[
h_0 = \left( \frac{\alpha}{2K_1 (K_2 C_D)^2} \right)^\frac{1}{p}.
\]

The second error bound in (3.10) is then a direct consequence of (3.15) with \( C_5 = C_4 K_2 C_D \).

(ii) Note from (2.13) and (3.3) that

\[
\lambda_{j,l} - \lambda_{j,l}^h = \frac{1}{l \pi} (w - w_h, \Delta(\eta^* s_{j,-l}))_{B_j(R;2R)}.
\]

Hence, (3.11) follows from the Cauchy-Schwarz inequality, Theorem 3.1(i), and Lemma 2.1 that

\[
|\lambda_{j,l} - \lambda_{j,l}^h| \leq \frac{1}{l \pi} \| w - w_h \| \| \Delta(\eta^* s_{j,-l}) \|_{B_j(R;2R)} \leq C_6 R^{\frac{1}{\pi} - 1} h^{\frac{p}{2}} \| f \|
\]

with \( C_6 = \frac{C_5 C_1}{l \pi} \).

(iii) It follows from (2.9) and (3.4) that

\[
u - u_h = (w - w_h) + \sum_{j=1}^{M} \sum_{l \in L_j} (\lambda_{j,l} - \lambda_{j,l}^h) \eta_{\rho_j} s_{j,l}.
\]

By using the triangle inequality, Lemma 2.1, (3.10), and (3.11), we have that

\[
\| u - u_h \|_1 \leq \| w - w_h \|_1 + \sum_{j=1}^{M} \sum_{l \in L_j} |\lambda_{j,l} - \lambda_{j,l}^h| \| \eta_{\rho_j} s_{j,l} \|_{1,B(\rho_j;R)} \leq C_4 h \| f \| + \sum_{j=1}^{M} \sum_{l \in L_j} C_6 \rho_j^{\frac{1}{\pi}} (C_2 \rho_j + C_3 R^{-1}) h^{1 + \frac{p}{2}} \| f \|.
\]

Therefore, the first inequality of (3.12) is valid with \( C_7 = C_4 + C_6 \sum_{j=1}^{M} \sum_{l \in L_j} \rho_j^{\frac{1}{\pi}} (C_2 \rho_j + C_3 R^{-1}) h^\frac{p}{2} \). In a similar fashion, by Lemma 2.1, (3.10), and (3.11), we may prove the validity of the second inequality of (3.12) with \( C_8 = C_5 + C_6 C_2 \sum_{j=1}^{M} \sum_{l \in L_j} \rho_j^{\frac{1}{\pi} + \frac{p}{2}} \). This completes the proof of the theorem.
4 Numerical Results

In this section, numerical results for Poisson equations with mixed boundary conditions are presented. One example is defined on the unit square and the other on a domain with re-entrance corner.

Example 1. Consider the Poisson equation in (1.1) with mixed boundary conditions on the unit square $\Omega = \{(x, y) \in \mathbb{R}^2 : 0 < x < 1, 0 < y < 1\}$. Homogeneous Neumann boundary is $\Gamma_N = \{(x, 0) \in \mathbb{R}^2 : 0 < x < 1/2\}$ and homogeneous Dirichlet boundary is $\Gamma_D = \partial \Omega \setminus \Gamma_N$ (see Fig.1(a)). This problem has a geometric singularity at boundary point $(1/2, 0)$, where the boundary conditions change from Dirichlet to Neumann with an internal angle $\omega = \pi$. More specifically, the corresponding singular function has the form

$$s = r^{\frac{1}{2}} \sin \left(\frac{\theta}{2}\right).$$

Let $\eta_2$ be the cut-off function defined in section 2 with $R = 1/4$ and choose the right-hand side function in (1.1) to be

$$f = -\sin(\pi x) \left[ -\pi^2 y^2 (y - 1) + 2(3y - 1) \right] - \Delta(\eta_2 s).$$

Then the exact solution of the underlying problem is

$$u = w + \eta_\rho s,$$

where $\eta_\rho$ is the cut-off function with $R = 1/4$ and $0 < \rho \leq 1$ and

$$w = \sin(\pi x) y^2 (y - 1) + (\eta_2 - \eta_\rho) s$$

is the regular part of the solution. Numerical results are presented in Table 1, 2, and 3, respectively, that confirm theoretical estimates.
Table 1. The discrete $L^2$-norm errors and the convergence rates for $w$.

<table>
<thead>
<tr>
<th>Mesh Size</th>
<th>$\rho = 1.00$</th>
<th>$\rho = 0.50$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$h = \frac{1}{8}$</td>
<td>2.3331e-02</td>
<td>2.0938e-02</td>
</tr>
<tr>
<td>$h = \frac{1}{16}$</td>
<td>8.5794e-03</td>
<td>6.8472e-03</td>
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<td>$h = \frac{1}{32}$</td>
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<td>2.1656e-03</td>
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<td>$h = \frac{1}{64}$</td>
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<td>5.9684e-04</td>
</tr>
<tr>
<td>$h = \frac{1}{128}$</td>
<td>1.7708e-04</td>
<td>1.5436e-04</td>
</tr>
</tbody>
</table>

Table 2. The discrete $H^1$ seminorm and the convergence rates for $w$.

<table>
<thead>
<tr>
<th>Mesh Size</th>
<th>$\rho = 1.00$</th>
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</thead>
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<tr>
<td>$h = \frac{1}{8}$</td>
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<td>3.2662e-01</td>
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<tr>
<td>$h = \frac{1}{16}$</td>
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<td>1.6750e-01</td>
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<td>$h = \frac{1}{64}$</td>
<td>1.4789e-02</td>
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<tr>
<td>$h = \frac{1}{128}$</td>
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</tbody>
</table>

Table 3. The absolute value errors and the convergence rates for $\lambda$.

<table>
<thead>
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<td>6.6797e-01</td>
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<td>$h = \frac{1}{128}$</td>
<td>3.4729e-03</td>
<td>4.3852e-03</td>
</tr>
</tbody>
</table>

Example 2. In this example, we consider a polygonal domain with a re-entrant corner (see Fig.1(b)):

$$\Omega = \{(x, y) \in \mathbb{R}^2 : -1 < x < 1, -1 < y < 1\} \setminus \bar{T},$$
where $T = \{(x, y) \in \mathbb{R}^2 : 0 < x < 1, -x < y < 0\}$. Homogeneous Neumann boundary is $\Gamma_N = \{(x, 0) \in \mathbb{R}^2 : 0 < x < 1, y = 0\}$ and homogeneous Dirichlet boundary is $\Gamma_D = \partial \Omega \setminus \Gamma_N$. At the origin, the internal angle is $\omega = \frac{7\pi}{4}$ and the boundary conditions change from Neumann to Dirichlet. Hence, there are two singular functions at the origin:

$$s_\frac{1}{2} = r^\frac{1}{2} \cos \frac{2\theta}{7} \quad \text{and} \quad s_\frac{3}{2} = r^\frac{1}{2} \cos \frac{6\theta}{7}.$$

Let $\eta_\rho$ be the cut-off function with $R = 1/4$ and $0 < \rho \leq 1$ and choose the right-hand side function in (1.1) to be

$$f = f_1 - \Delta(\eta_2(s_\frac{1}{2} + s_\frac{3}{2})),$$

where

$$f_1 = \begin{cases} 
-2(y - 1)y^2 - (x^2 - 1)(6y - 2) & \text{if } y \geq 0, \\
-2(y + 1)y^2 - (x + 1)(6xy + 2x + 12y^2 + 6y) & \text{if } y < 0
\end{cases}.$$

The exact solution of the underlying problem is then

$$u = w + \eta_\rho s_\frac{1}{2} + \eta_\rho s_\frac{3}{2}$$

where the regular part $w$ is given by

$$w = (\eta_2 - \eta_\rho)(s_\frac{1}{2} + s_\frac{3}{2}) + \begin{cases} 
(x^2 - 1)(y - 1)y^2 & \text{if } y \geq 0, \\
(x + 1)(y + 1)(x + y)y^2 & \text{if } y \leq 0
\end{cases}.$$

Note that the function $w$ is in $H^2(\Omega)$, but not in $H^3(\Omega)$. Numerical results for the discretization accuracy of the finite element approximation to $w$ are given in Tables 4 and 5. Results for the stress intensity factors are contained in Tables 6 and 7.

<table>
<thead>
<tr>
<th>Mesh Size</th>
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<tbody>
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<td>RATE</td>
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<td>1.8582</td>
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</table>

Table 4. The discrete $L^2$-norm errors and the convergence rates for $w$. 

13
<table>
<thead>
<tr>
<th>$\rho = 1.00$</th>
<th>$\rho = 0.50$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$H^1$-norm</td>
<td>Rate</td>
</tr>
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Table 5. The discrete $H^1$ seminorm and the convergence rates for $w$.

<table>
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<th>$\rho = 0.50$</th>
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</thead>
<tbody>
<tr>
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</tr>
<tr>
<td>$h = \frac{1}{8}$</td>
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</table>

Table 6. The absolute value errors and the convergence rates for $\lambda^{1/2}$.

<table>
<thead>
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<th>$\rho = 1.00$</th>
<th>$\rho = 0.50$</th>
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<tbody>
<tr>
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Table 7. The absolute value errors and the convergence rates for $\lambda^{3/2}$.

References


