Part I. Weak Solution, Characterists, Entropy Solutions

① Want to solve initial value problem (IVP) of scalar conservation laws

$$\begin{cases} u_t + f(u)_x = 0 \\ u(x,0) = u_0(x) \end{cases}$$

Example: Burger's Equation

$$u_t + \left(\frac{u^2}{2}\right)_x = 0$$

Assume $u(x,t)$ is smooth and consider a curve $x(t)$ defined as

$$\begin{cases} x'(t) = f'(u(x(t),t)) \\ x(0) = x_0 \end{cases}$$

$$\frac{\mathrm{d}u(x(t),t)}{\mathrm{d}t} = u_x x'(t) + u_t = u_x f'(u(x(t),t) + u_t = f(u)_x + u_t = 0.$$

$\Rightarrow$ $u(x(t),t)$ is a constant C Solution along this curve is Constant

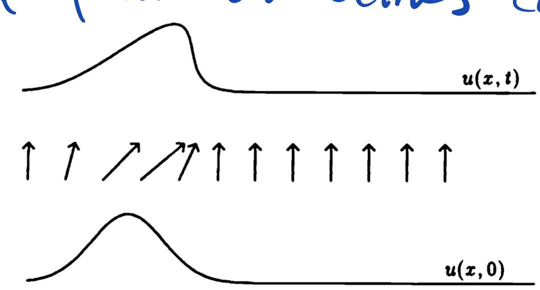
$\Rightarrow$ $x'(t) = f'(u(x,t)) = f'(C)$

$\Rightarrow$ This curve is a line Characteristic Line

② Consider characteristic lines for

$$\begin{cases} u_t + \left(\frac{u^2}{2}\right)_x = 0, \\ u(x,0) = \sin(x). \end{cases}$$

Then we find two lines can intersect.



$\Rightarrow$ shocks / discontinuities

$\Rightarrow f(u)_x$ is not even well defined in weak sense

Need to define weak solutions

Two equivalent definitions of weak solutions:

1) $$\frac{\mathrm{d}}{\mathrm{d}t}\int_a^b u(x,t)\mathrm{d}x = f(u(a,t)) - f(u(b,t))$$

holds for almost all intervals (a,b)

2) $$-\int_0^\infty \int_{-\infty}^\infty (u\varphi_t + f(u)\varphi_x)\mathrm{d}x\,\mathrm{d}t - \int_{-\infty}^\infty u^0(x)\varphi(x,0)\mathrm{d}x = 0,$$

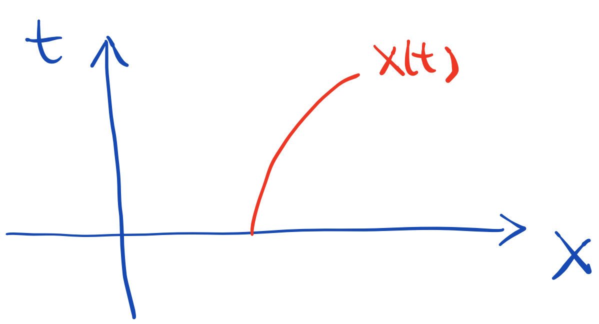
$\forall \varphi \in C_0^1(\mathbb{R}^2)$

③ Weak Sol $\Rightarrow$ Rankine-Hugoniot Jump Condition

Assume $u(x,t)$ is piecewise smooth w.r.t. x

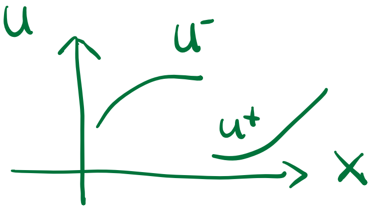
with two pieces

and shock/jump location is $x(t)$



$$u^- := u(x(t^-),t)$$
$$u^+ := u(x(t^+),t)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_a^b u(x,t)\mathrm{d}x = f(u(a,t)) - f(u(b,t)) \Rightarrow x'(t) = \frac{f(u^+)-f(u^-)}{u^+-u^-}$$



$$
\begin{aligned}
0 &= \frac{\mathrm{d}}{\mathrm{d}t}\int_a^b u(x,t)\mathrm{d}x + f(u(b,t)) - f(u(a,t))\\
&= \frac{\mathrm{d}}{\mathrm{d}t}\left[\int_a^{x(t)} u(x,t)\mathrm{d}x + \int_{x(t)}^b u(x,t)\mathrm{d}x\right] + f(u(b,t)) - f(u(a,t))\\
&= u(x(t^-),t)x'(t) + \int_a^{x(t)} u_t(x,t)\mathrm{d}x - u(x(t^+),t)x'(t) + \int_{x(t)}^b u_t(x,t) + f(u(b,t)) - f(u(a,t))\\
&= u(x(t^-),t)x'(t) - \int_a^{x(t)} f(u)_x\mathrm{d}x - u(x(t^+),t)x'(t) + \int_{x(t)}^b f(u)_x\mathrm{d}x + f(u(b,t)) - f(u(a,t))\\
&= u(x(t^-),\, t)x'(t) - f(u(x(t^-),\, t)) + f(u(a,\, t)) - u(x(t^+),\, t)x'(t) - f(u(b,\, t)) - f(u(x(t^+),\, t)) + f(u(b,\\
&\quad t)) - f(u(a,t))\\
&= u(x(t^-),t)x'(t) - f(u(x(t^-),t) - u(x(t^+),t)x'(t) + f(u(x(t^+),t).
\end{aligned}
$$

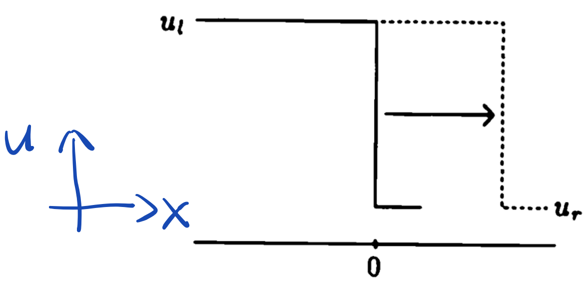
RH Jump Condition $\Longleftrightarrow$ Weak Sol

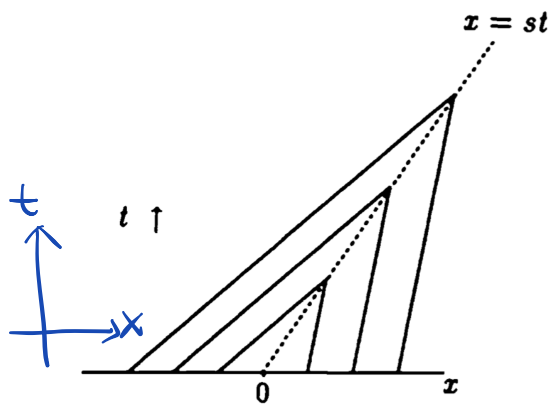
If u is piecewise C^1 and is discontinuous only along isoated curves, and if u satisfies the PDE when it is C^1, and the Rankine-Hugoniot (RH) condition along all discontinuous cruves, then u is a weak solution of

Example 1. Consider the following *Riemann problem*:

$$
\begin{cases}
u_t + \left(\frac{u^2}{2}\right)_x = 0\\
u(x,0) = \left\{\begin{array}{ll} 1 & x<0,\\ -1 & x>0.\end{array}\right.
\end{cases}
$$

The IC is just propagated in time to form a weak solution. (a *shock*)





Example 2. Now flip the initial conditions:

$$
\begin{cases}
u_t + \left(\frac{u^2}{2}\right)_x = 0\\
u(x,0) = \left\{\begin{array}{ll} -1 & x<0,\\ 1 & x>0.\end{array}\right.
\end{cases}
$$

The propagated ICs also form a weak solution. But consider

$$
u(x,t) = \begin{cases} -1 & x \leqslant -t,\\ x/t & -t<x<t,\\ 1 & x>t.\end{cases}
$$

This is also a weak solution. (a *rarefaction wave*)

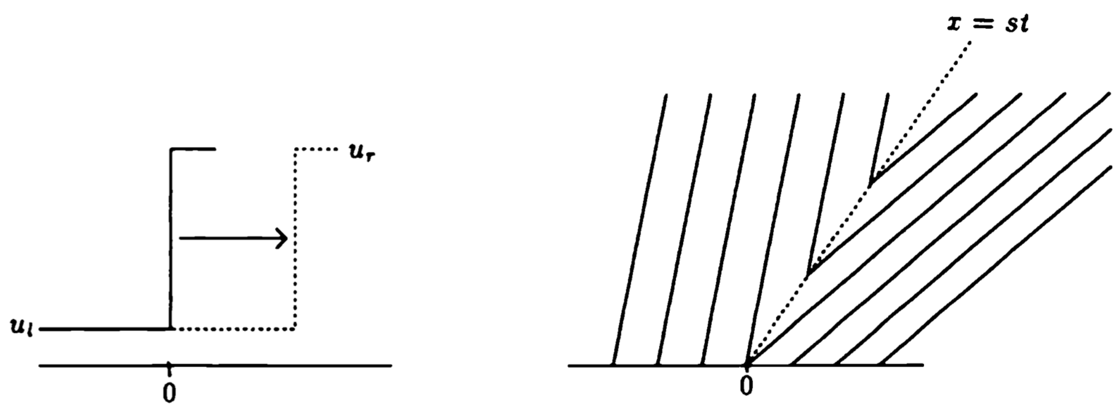


Figure 3.9. Entropy-violating shock.

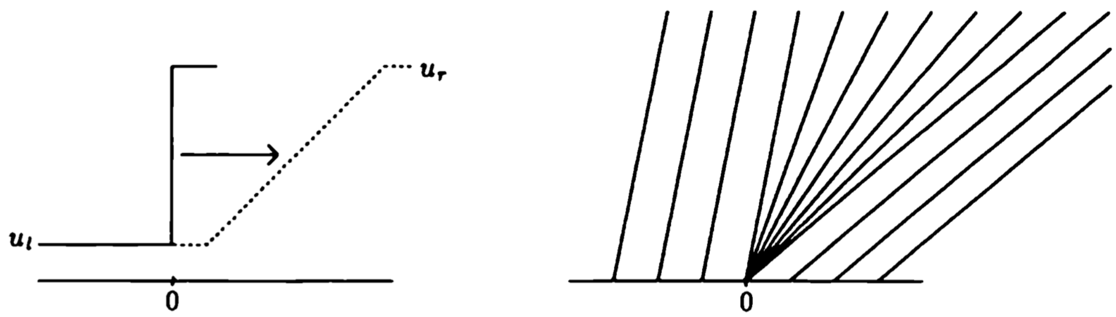


Figure 3.10. Rarefaction wave.

④ So weak sol is not unique, and we want the unique physical sol (vanishing viscosity sol)

<u>Def</u> (vanishing viscosity sol) $\varepsilon > 0$

$$\begin{cases} u_t^\varepsilon + f(u^\varepsilon)_x = \varepsilon u^\varepsilon_{xx} \\ u^\varepsilon(x,0) = u_0(x) \end{cases}$$

$\lim\limits_{\varepsilon \to 0} u^\varepsilon(x,t)$ is called vanishing viscosity sol.

Vanishing Viscosity $\Rightarrow$ Entropy Solution

Mathematical Enptropy Function is any convex $U(u)$

i.e., $U''(u) \geq 0$

Entropy Flux : $F'(u) = U'(u) f'(u)$

Pick a function $U(u)$ called the *entropy function* if $U''(u) \geqslant 0$, i.e. if it is convex. Then multiply the conservation law with viscosity by $U'(u^\varepsilon)$:

$$\begin{aligned} U'(u^\varepsilon)(u_t^\varepsilon + f(u^\varepsilon)_x) &= \varepsilon U'(u^\varepsilon) u_{x,x}^\varepsilon \\ U(u^\varepsilon)_t + F(u^\varepsilon)_x &= \varepsilon\left[(U'(u^\varepsilon)u_x^\varepsilon)_x - U''(u^\varepsilon)(u_x^\varepsilon)^2\right] \\ U(u^\varepsilon)_t + F(u^\varepsilon)_x &\leqslant \varepsilon(U'(u^\varepsilon)u_x^\varepsilon)_x \end{aligned}$$

where

$$F(u) = \int^u U'(v) f'(v) \mathrm{d}v \quad \Rightarrow \quad F'(u) = U'(u) f'(u).$$

To support our argument as $\varepsilon \to 0$, once again take a test function $\varphi \in C_0^2(\mathbb{R} \times \mathbb{R}^+)$, $\varphi \geqslant 0$.

$$\begin{aligned} \int_0^\infty \int_{-\infty}^\infty (U(u^\varepsilon)_t + F(u^\varepsilon)_x)\varphi \,\mathrm{d}x\,\mathrm{d}t &\leqslant \varepsilon \int_0^\infty \int_{-\infty}^\infty (U'(u^\varepsilon)u_x^\varepsilon)_x \varphi \mathrm{d}x\,\mathrm{d}t \\ \Rightarrow \int_0^\infty \int_{-\infty}^\infty U(u^\varepsilon)\varphi_t + F(u^\varepsilon)\varphi_x \,\mathrm{d}x\,\mathrm{d}t &\geqslant \varepsilon \int_0^\infty \int_{-\infty}^\infty U'(u^\varepsilon)u_x^\varepsilon \varphi_x \mathrm{d}x\,\mathrm{d}t \\ &= -\varepsilon \int_0^\infty \int_{-\infty}^\infty U(u^\varepsilon)\varphi_{x,x} \mathrm{d}x\,\mathrm{d}t \end{aligned}$$

By Dominated Convergence Theorem, we take $\varepsilon \to 0$

We get the *entropy inequality*

$$\int_0^\infty \int_{-\infty}^\infty U(u)\varphi_t + F(u)\varphi_x \,\mathrm{d}x\,\mathrm{d}t \geqslant 0.$$

Sometimes written as $U(u)_t + F(u)_x \geqslant 0$

We want to solve $u_t + f(u)_x = 0$

with $U(u)_t + F(u)_x \geqslant 0$ enforced

for any entropy pair (U, F).

1) This is extremely difficult for general $f(u)$!

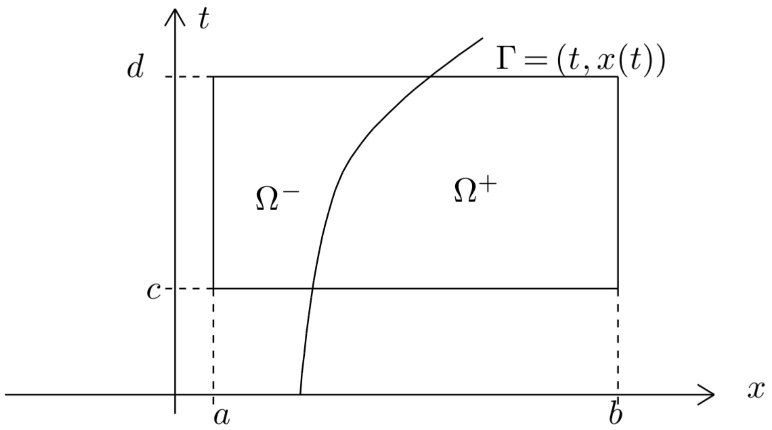
2) If $f(u)$ is convex w.r.t. u, then

enforcing one entropy pair (U, F) is enough.

Definition 3. *A conservation law is called* genuinely nonlinear *iff* $f''(u) \neq 0$. *If* $f''(u) > 0$, *it is called* convex, *if* $f''(u) < 0$ *it is called* concave.

Shocks must appear for genuinely nonlinear conservation laws under periodic or compactly supported initial conditions.

⑤ Entropy Condition / Inequality



Consider a box containing the support of a test function $\varphi \in C_c^\infty(\mathbb{R} \times \mathbb{R}^+)$ and let $u(x,t)$ be piecewise C^1 with one discontinuity along $(t, x(t))$.

Then consider

$$\begin{aligned}
0 &\leqslant -\int_c^d \int_a^b (U(u)\varphi_t + F(u)\varphi_x)\mathrm{d}x\mathrm{d}t \\
&= -\int_c^d \int_a^{x(t)} \underbrace{(U(u)\varphi_t + F(u)\varphi_x)}_{(U,F)^T \cdot \nabla\varphi}\mathrm{d}x\mathrm{d}t - \int_c^d \int_{x(t)}^b (U(u)\varphi_t + F(u)\varphi_x)\mathrm{d}x\mathrm{d}t \\
&= \int_c^d \int_a^{x(t)} \underbrace{(U(u)_t + F(u)_x)}_{=0}\varphi\mathrm{d}x\mathrm{d}t - \int_{\partial\Omega^-} \varphi(U(u), F(u)) \cdot \boldsymbol{n}\mathrm{d}s - \int_{\partial\Omega^+} \varphi(U(u), F(u)) \cdot \boldsymbol{n}\mathrm{d}s \\
&= \int_\Gamma \varphi \frac{x'(t)U(u^-) - F(u^-)}{\sqrt{1 + (x'(t))^2}}\mathrm{d}s - \int_\Gamma \varphi \frac{x'(t)U(u^+) - F(u^+)}{\sqrt{1 + x'(t)^2}}\mathrm{d}s \\
&= \int_\Gamma \frac{\varphi}{\sqrt{1 + x'(t)^2}}\left[x'(t)(U(u^-) - U(u^+)) - (F(u^-) - F(u^+))\right]\mathrm{d}s.
\end{aligned}$$

We obtain

$$x'(t)(U(u^-) - U(u^+)) - (F(u^-) - F(u^+)) \leqslant 0.$$

If we introduce the notation $[\![f]\!] := f(u^+) - f(u^-)$, then this condition becomes

$$x'(t)[\![U]\!] \geqslant [\![F]\!].$$

Oleinik entropy condition: For all u between u^- and u^+, we need to have

$$\frac{f(u)-f(u^-)}{u-u^-} \geqslant \underbrace{x'(t)}_{s} \geqslant \frac{f(u)-f(u^+)}{u-u^+},$$

where s is the shock speed, known from the Rankine-Hugoniot condition.

Lax's entropy condition:

$$f'(u^-) > s > f'(u^+).$$

It is easy to see that the Oleinik condition implies Lax's condition. Unfortunately, the converse does not hold. Lax's entropy condition does not guarantee uniqueness–but it is a necessary condition. However, if $f''(u) \gtrless 0$ uniformly (i.e. the conservation law is genuinely nonlinear), *then* Lax's entropy condition is sufficient for u to be the entropy solution.

For $f'(u) > 0$, Lax's condition becomes even simpler. Consider

$$f'(u^-) \geqslant s = \frac{[\![f(u)]\!]}{[\![u]\!]} \geqslant f'(u^+)$$

and note that $f'(u)$ is monotonically increasing, such that the middle part is automatically satisfied. Thus, Lax's condition becomes

$$f'(u^-) \geqslant f'(u^+).$$

6 L^1- contraction and TVD

1) **Theorem 4.** *The solutions to*

$$\begin{cases} u_t^\varepsilon + f(u^\varepsilon)_x = \varepsilon u_{x,x}^\varepsilon, \\ u^\varepsilon(x,0) = u^0(x) \end{cases}$$

are L^1-contractive. I.e. let v^ε be the solution of

$$\begin{cases} v_t^\varepsilon + f(v^\varepsilon)_x = \varepsilon v_{x,x}^\varepsilon, \\ v^\varepsilon(x,0) = v^0(x). \end{cases}$$

Then

$$\|u^\varepsilon(\cdot,t) - v^\varepsilon(\cdot,t)\|_{L^1} \leqslant \|u^0 - v^0\|_{L^1}.$$

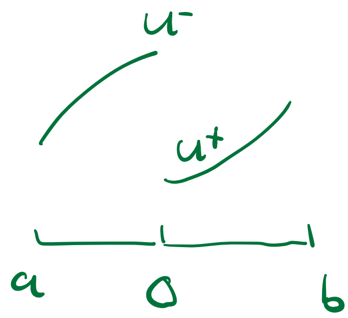
2) The entropy solution has a non-increasing total variation.

$$\mathrm{TV}(u) := \sup_h \int \left|\frac{u(x+h)-u(x)}{h}\right| \mathrm{d}x.$$

$$\mathrm{TV}(u(\cdot,t)) \leqslant \mathrm{TV}(u^0),$$

a) If $u(x)$ is smooth, $TV(u) = \int |u'(x)|\, dx$

b) If $u(x)$ is is piecewise smooth



$$u(x) = \begin{cases} u_l(x), & x<0 \\ u_r(x), & x>0 \end{cases}$$

then $TV(u) = \int_a^0 |u_l'(x)|\,dx + \int_0^b |u_r'(x)|\,dx + |u^+ - u^-|$

3) the analytic solution satisfies a *maximum principle*, i.e.

$$\min_x u^0(x) \leqslant u(\xi, t) \leqslant \max_x u^0(x).$$

Part II Schemes

① Conservative schemes

Definition 5. *A scheme to solve conservation laws is called* conservative *iff it can be written as*

$$u_j^{n+1} = u_j^n - \frac{\Delta t}{\Delta x}\left[\hat{f}_{j+1/2} - \hat{f}_{j-1/2}\right],$$

where $\hat{f}$ *is*

1. *Lipschitz continuous,*
2. $\hat{f}(u, \cdots, u) = f(u)$ (consistency).

Example:

$u_t + a\,u_x = 0$, we wrote down an *upwind scheme*:

$$u_j^{n+1} = u_j^n - a \cdot \frac{\Delta t}{\Delta x}(u_j^n - u_{j-1}^n).$$

$$\hat{f}_{j+\frac{1}{2}} = \hat{f}(u_j, u_{j+1}) = f(u_j) = a u_j$$

$$\hat{f}_{j-\frac{1}{2}} = \hat{f}(u_{j-1}, u_j) = f(u_{j-1}) = a u_{j-1}$$

Bi if
y

Theorem 6. **(Lax-Wendroff)** *If the solution* $\{u_j^n\}$ *to a conservative scheme converges (as* $\Delta t, \Delta x \to 0$*) boundedly a.e. to a function* $u(x,t)$*, then* u *is a weak solution of the conservation law.*

Proof. Let $\varphi_j^n = \varphi(x_j, t^n)$ for $\varphi \in C_0^1$. Then

$$\begin{aligned} 0 &= \sum_n \sum_j \left(\frac{u_j^{n+1} - u_j^n}{\Delta t} + \frac{\hat{f}_{j+1/2} - \hat{f}_{j-1/2}}{\Delta x} \right) \varphi_j^n \Delta x \Delta t \\ &= -\sum_n \sum_j \left(\frac{\varphi_j^n - \varphi_j^{n-1}}{\Delta t} u_j^n + \frac{\varphi_j^n - \varphi_{j-1}^n}{\Delta x} \hat{f}_{j+1/2} \right) \Delta x \Delta t \\ &\overset{\text{DCT,Conservativity}}{\longrightarrow} \int_0^\infty \int_{-\infty}^\infty (\varphi_t u + \varphi_x f(u)) \mathrm{d}x \, \mathrm{d}t = 0. \end{aligned}$$

□

Above, we used partial summation:

$$\sum_{j=j_1}^{j_2} a_j (b_j - b_{j-1}) = -\sum_{j=j_1}^{j_2} (a_{j+1} - a_j) b_j - a_{j_1} b_{j-1} + a_{j_2} b_{j_2}.$$

Remark: A conservative scheme may not converge

e.g., take $\hat{f}_{j+\frac{1}{2}} = \dfrac{f(u_j) + f(u_{j+1})}{2}$ for Burger's

② Riemann Problem

$$\begin{cases} u_t + f(u)_x = 0 \\ u(x,0) = \begin{cases} u_l, & x<0 \\ u_r, & x>0 \end{cases} \end{cases}$$

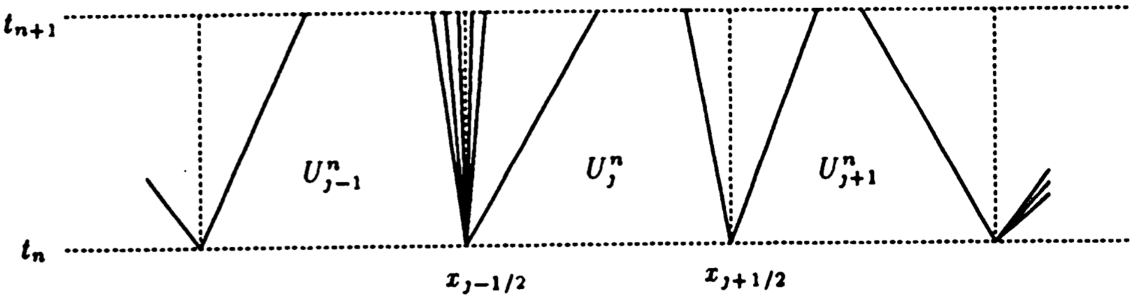
u_l, u_r are constants

Riemann solution $u(x,t)$ is a function of only one variable $\xi = x/t$.

Self-Similary Sol

③ Godunov's Scheme

Consider mesh with intervals $I_j = [x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}]$



0) Piece-wise constant $\bar{u}_j^n$ in each interval I_j

first order accurate approximation

Ideas

1) Assume exact sol to Riemann Problem is given

2) Assume Δt is not too Large so that characterics from different Riemann Problems do not intersect, e.g., no intersection of shocks

1) shock speed is $x'(t) = \dfrac{f(u^+) - f(u^-)}{u^+ - u^-} = f'(u)$ some u

2) Any non-intersecting characteristic lines has speed

$$x'(t) = f'(u)$$

So $\Delta t \max\limits_u |f'(u)| \leq \frac{1}{2} \Delta x$ is sufficient

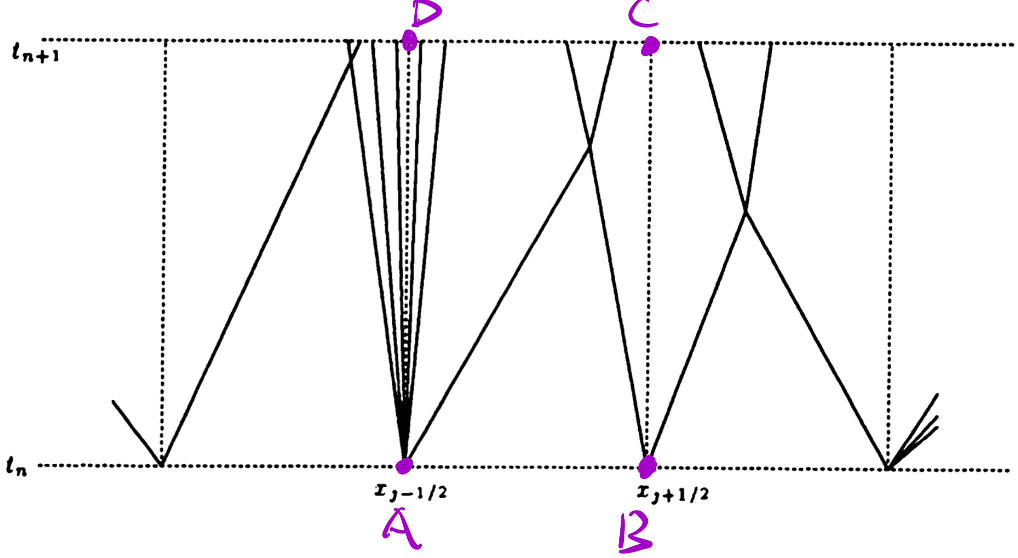
And we only need to solve Riemann Problems

Just need $f(u(x_{j+\frac{1}{2}}, t))$ stay constant

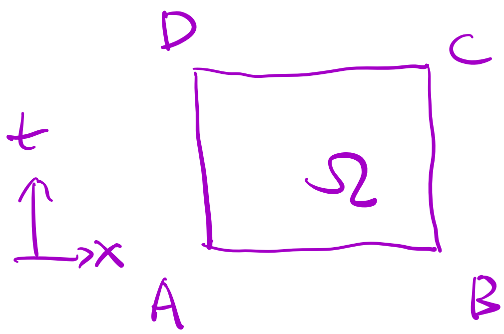
for $t \in [t_n, t_{n+1}]$ ⇐ Self-similar of Riemann

$\Delta t \max\limits_u |f'(u)| \leq \Delta x$ is sufficient

to ensure the characteristic line passing $(x_{j+\frac{1}{2}}, t_n)$ and $(x_{j-\frac{1}{2}}, t_{n+1})$ do not intersect others



4)



Let $\tilde{u}(x,t)$ be exact sol to $\begin{cases} u_t + f(u)_x = 0 \\ u(x,t_n) \text{ is piecewise constant} \end{cases}$

Weak Sol $\dfrac{\mathrm{d}}{\mathrm{d}t}\displaystyle\int_a^b u(x,t)\mathrm{d}x = f(u(a,t)) - f(u(b,t)) \Rightarrow$

Or equivalently

$$\int_{t^n}^{t^{n+1}}\int_{x_{j-1/2}}^{x_{j+1/2}}(u_t + f(u)_x)\mathrm{d}x\,\mathrm{d}t = 0$$

$$\frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}} u^{n+1}\,\mathrm{d}x - \frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}} u^n\,\mathrm{d}x + \frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} f(u_{j+1/2})\,\mathrm{d}x - \frac{1}{\Delta x}\int_{t^n}^{t^{n+1}} f(u_{j-1/2})\mathrm{d}x = 0.$$

$$(*)\quad \int_{x_{j-1/2}}^{x_{j+1/2}} \tilde{u}^n(x,t_{n+1})\,dx = \int_{x_{j-1/2}}^{x_{j+1/2}} \tilde{u}^n(x,t_n)\,dx + \int_{t_n}^{t_{n+1}} f(\tilde{u}^n(x_{j-1/2},t))\,dt - \int_{t_n}^{t_{n+1}} f(\tilde{u}^n(x_{j+1/2},t))\,dt.$$

Let $\begin{cases} u_j^n = \frac{1}{\Delta x}\int_{I_j}\tilde{u}^n(x,t_n)\,dx \\ \hat{f}(u_j^n,u_{j+1}^n) = \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} f(\tilde{u}^n(x_{j+\frac{1}{2}},t))\,dt \end{cases}$

Then (*) for $f''(u)>0$ is

$$\left\{\begin{array}{l} u_j^{n+1} = u_j^n - \frac{\Delta t}{\Delta x}\left[\hat{f}(u_j^n, u_j^{n+1}) - \hat{f}(u_{j-1}^n, u_j^n)\right] \\ \hat{f}_{j+1/2} = \begin{cases} \min_{u_j \leqslant u \leqslant u_{j+1}} f(u) & u_j < u_{j+1}, \\ \max_{u_j \leqslant u \leqslant u_{j+1}} f(u) & u_j \geqslant u_{j+1}. \end{cases} \\ \frac{\Delta t}{\Delta x} \max_u |f'(u)| \leqslant 1 \end{array}\right.$$

② Monotone Schemes

Definition 8. *A scheme*

$$\begin{aligned} u_j^{n+1} &= u_j^n - \lambda(\hat{f}(u_{j-p}, \ldots, u_{j+q}) - \hat{f}(u_{j-p-1}, \ldots, u_{j+q-1})) \\ &\equiv G(u_{j-p-1}, \ldots, u_{j+q}) \end{aligned}$$

is called a montone scheme *if G is a monotonically nondecreasing function $G(\uparrow, \uparrow, \ldots, \uparrow)$ of each argument.*

In the special case of 3-point schemes

$$\hat{f}(u_j, u_{j+1}) \qquad \lambda = \frac{\Delta t}{\Delta x}$$

the scheme is a monotone if $f(\uparrow, \downarrow)$ plus a restriction on λ:

$$G(u_{j-1}, u_j, u_{j+1}) = u_j - \lambda[\hat{f}(u_j, u_{j+1}) - \hat{f}(u_{j-1}, u_j)].$$

Clearly, if $\hat{f}(\uparrow, \downarrow)$, then $G(\uparrow, ?, \uparrow)$. To clean up the second argument, consider

$$\frac{\partial G}{\partial u_j} = 1 - \lambda[\underbrace{\hat{f}_1 - \hat{f}_2}_{\geqslant 0}] \geqslant 0.$$

If $\lambda(\hat{f}_1 - \hat{f}_2) \leqslant 1$, then $G(\uparrow, \uparrow, \uparrow)$.

Examples: The Lax-Friedrichs flux is monotone:

$$\begin{aligned} \hat{f}^{\,\mathrm{LF}}(u_j, u_{j+1}) &= \frac{1}{2}[f(u_j) + f(u_{j+1}) - \alpha(u_{j+1} - u_j)] \quad \text{for} \quad \alpha = \max_u |f'(u)|, \\ \hat{f}_1^{\mathrm{LF}} &= \frac{1}{2}[f'(u_j) + \alpha] \geqslant 0, \\ \hat{f}_2^{\mathrm{LF}} &= \frac{1}{2}[f'(u_{j+1}) + \alpha] \leqslant 0. \end{aligned}$$

$$\frac{\partial \hat{f}(u_j, u_{j+1})}{\partial u_j} = \frac{\alpha + f'(u_j)}{2}; \quad \frac{\partial \hat{f}(u_j, u_{j+1})}{\partial u_{j+1}} = \frac{-\alpha + f'(u_{j+1})}{2}$$

$$u_j^{n+1} = u_j^n - \frac{\Delta t}{\Delta x}\left[\hat{f}(u_j, u_{j+1}) - \hat{f}(u_{j-1}, u_j)\right]$$

$$= G(u_{j-1}, u_j, u_{j+1})$$

$$\frac{\partial G}{\partial u_{j-1}} = \frac{\Delta t}{\Delta x}\frac{\alpha + f'(u_{j-1})}{2} \geqslant 0, \quad \frac{\partial G}{\partial u_{j+1}} = \frac{\Delta t}{\Delta x}\frac{\alpha - f'(u_{j+1})}{2} \geqslant 0$$

$$\frac{\partial G}{\partial u_j} = 1 - \frac{\Delta t}{\Delta x}\alpha \geqslant 0 \quad \Leftarrow \quad \frac{\Delta t}{\Delta x}\alpha \leq 1$$

CFL condition

Theorem 9. *Good properties of monotone schemes:*

1. $u_j \leqslant v_j$ *for all* j *("*$u \leqslant v$*") implies* $G(u)_j \leqslant G(v)_j$ *for all* j.
2. Local maximum principle:

$$\min_{i \in \text{stencil around } j} u_i \leqslant G(u)_j \leqslant \max_{i \in \text{stencil around } j} u_i.$$

3. L^1*-contraction: (this was already obtained for the PDE)*

$$\|G(u) - G(v)\|_{L^1} \leqslant \|u - v\|.$$

4. *This immediately implies the* Total Variation Diminishing *(TVD) property:*

$$\|G(u)\|_{\mathrm{BV}} \leqslant \|u\|_{\mathrm{BV}}.$$

Proof. 1 is just the definition.

2. Fix j. Take

$$v_i = \begin{cases} \max_{k \in \text{stencil arond } i} u_k & \text{if } i \in \text{stencil around } j, \\ u_i & \text{otherwise.} \end{cases}$$

Then clearly $u_i \leqslant v_i$ for all i, so that

$$G(u)_j \leqslant G(v)_j = v_j = \max_{i \in \text{stencil around } j} u_i.$$

Other way around runs in an analogous fashion.

3. Define

$$a \vee b = \max(a, b), \quad a \wedge b = \min(a, b), \quad a^+ = a \wedge 0, \quad a^- = a \vee 0.$$

Then let

$$w_j := u_j \vee v_j = v_j + (u_j - v_j)^+. \quad (*)$$

We have

$$G(u)_j \leqslant G(w)_j \geqslant G(v)_j \quad \forall j$$

by property 1. Then

$$G(w)_j - G(v)_j \geqslant \begin{cases} 0 & \forall j, \\ G(u_j) - G(v_j) & \forall j. \end{cases}$$

Thus

$$G(w)_j - G(v)_j \geqslant (G(u)_j - G(v)_j)^+.$$

Therefore

$$\sum_j (G(u)_j - G(v_j))^+ \leqslant \sum_j (G(w)_j - G(v))_j \overset{(**)}{=} \sum_j w_j - v_j \overset{(*)}{=} \sum_j (u_j - v_j)^+.$$

because we are treating a *conservation* law, meaning

$$\sum_j u_j^{n+1} = \sum_j u_j^n, \quad (**)$$

which holds for *conservative schemes*. (Why?) Also consider

$$\begin{aligned} \sum_j |G(u)_j - G(v)_j| &= \sum_j (G(u)_j - G(v)_j)^+ + \sum_j (G(u)_j - G(v)_j)^- \\ &\leqslant \sum_j (u_j - v_j)^+ + \sum_j (v_j - u_j)^+ \\ &= \sum_j |u_j - v_j|. \end{aligned}$$

(This is also called the *Crandall-Tartar lemma*.)

4: Take $v_j = u_{j+1}$ in 3. □

Theorem Discrete Entropy inequality can be proven for monotone schemes.

Convergence:

① Monotone $\Rightarrow$ TVD $\Rightarrow$ Compactness $\Rightarrow$ converging subsequence as $\Delta x, \Delta t \to 0$

② Lax-Wendroff $\Rightarrow$ weak sol

③ Discrete Entropy inequality $\Rightarrow$ entropy sol.