Theorem 1 (Stone-Weierstrass). Let $A$ be an algebra as a subset of $C(X)$ where $X$ is a compact space. If $A$ separates points in $X$ and contains the constant functions, then $\overline{A} = C(X)$ in the uniform metric $\rho(f,g) = \| f - g \|$ where $\| f \| = \max_{x \in X} |f(x)|$.

We do some foundational works before proving Theorem 1.

Proposition 2. Let $L$ be a sublattice of $C(X)$ where $X$ is compact. Suppose the function $h$ defined by $h(x) = \inf_{f \in L} f(x)$ is continuous and finite, then given any $\epsilon > 0$, there is $g \in L$ such that $0 \leq g(x) - h(x) < \epsilon/3$.

Proof. Let $\epsilon > 0$ and let $x \in X$. By the definition of $h$, there is $f_x \in L$ such that $0 \leq f_x(x) - h(x) < \epsilon/3$. But both $f_x$ and $h$ are continuous. So there is open $O_x$ containing $x$ such that if $y \in O_x$, then $|f_x(y) - f_x(x)| < \epsilon/3$ and $|h(x) - h(y)| < \epsilon/3$. Hence, for each $y \in O_x$, we have $0 \leq f_x(y) - h(y) = |f_x(y) - h(y)| \leq |f_x(x) - h(x)| + |f_x(h) - h(x)| < \epsilon/3$. We have for each $x \in X$ the associated $f_x$ and $O_x$. So $\{O_x\}_{x \in X}$ is an open covering of $X$. By compactness, there is a finite subcovering $\{O_{x_1}, \ldots, O_{x_N}\}$ of $X$. Let $g = f_{x_1} \wedge \cdots \wedge f_{x_N}$. So $g \in L$. And we know that for any $y \in X$, there is $O_x$ containing $y$ such that $0 \leq g(y) - h(y) \leq f_x(y) - h(y) < \epsilon$, as desired. □

Remark:
1. We don’t use uniform continuity in the proof but to use the definition of compactness directly.
2. In the proof of the Proposition 3 we will first have a continuous function $h$ and then show that $h$ can be acquired by the above means ($h$ is the pointwise infimum of functions in a subsublattice) so as to get the desired approximation property.

Proposition 3. Let $X$ be a compact space and $L$ a sublattice of $C(X)$ such that it has the following two properties: (i) $L$ separates points and (ii) for any $c \in \mathbb{R}$ and $f \in L$ we have $c + f \in L$ and $cf \in L$.

Then for any $h \in C(X)$ and any $\epsilon > 0$, there is $g \in L$ such that for all $x \in X$, $0 \leq g(x) - h(x) < \epsilon$ and thus $\overline{L} = C(X)$.

We prove two lemmas first.

Lemma 4. Let $L \subset C(X)$. If $f$ has property (i) and (ii), then given any $a, b \in \mathbb{R}$ and distinct $x, y \in X$, there is $f \in L$ such that $f(x) = a$ and $f(y) = b$.

Proof. Since $L$ separates points, there is $g \in L$ such that $g(x) - g(y) \neq 0$. So $f$ defined by

$$f(t) = \frac{a(g(t) - g(y)) + b(g(x) - g(t))}{g(x) - g(y)}$$

is well-defined and $f \in L$ by property (i) and (ii). Notice that $f(x) = a$ and $f(y) = b$. □

Lemma 5. Let $L$ be a sublattice of $C(X)$ where $X$ is compact and $L$ satisfies (i) and (ii). Then given $a, b \in \mathbb{R}$ with $a \leq b$, $F$ a closed subset of $X$ and $p \notin F$, there is $f \in L$ such that $f \geq a$, $f(p) = a$ and $f(x) > b$ for all $x \in F$.

Proof. For any $x \in F$, $x \neq p$. So by Lemma 4, there is $f_x \in L$ such that $f_x(p) = a$ and $f_x(x) = b + 1$. Notice that since $f_x$ is continuous, there is open $O_x$ containing $x$ such that for all $y \in O_x$, $f_x(y) - f_x(x) \geq -\frac{1}{2}$ and hence $f_x(y) \geq f_x(x) - \frac{1}{2} = b + \frac{1}{2} > b$. For each $x \in F$, let $O_x$ and $f_x$ be the
associated open set and function satisfying the previous condition. Then \( \{O_x\}_{x \in F} \) covers \( F \). But \( F \) is compact being a closed subset of a compact set. So there is a finite subcovering \( \{O_x, \ldots, O_{x_N}\} \) of \( F \). Let \( g = f_{x_1} \lor \cdots \lor f_{x_N} \). So \( g(p) = a \) and for any \( y \in F \), there is \( O_x \) containing \( y \) so that \( g(y) \geq f_x(y) > b \). Finally, let \( f = a \lor g \) and this is the desired function (\( f \geq a, f(p) = a, f > b \) on \( F \)).

Now we prove Proposition 3.

**Proof.** We want to show \( \overline{L} = C(X) \). Given any \( h \in C(X) \), let \( l = \{f \in L : f \geq h\} \). If we can show for any \( p \in X \), \( h = \inf_{f \in l} f(p) \), by Proposition 2, given \( \epsilon > 0 \) there is \( g \in l \) and thus in \( L \) such that \( p(g, h) < \epsilon \) which shows \( h \in \overline{L} \).

We prove this by definition. Let \( p \in X \). Let \( \epsilon > 0 \). We know \( X \) is compact. Let \( m \) be the maximum of \( h \) on \( X \). Let \( M = \max\{m, h(p) + \epsilon\} \geq h(p) + \epsilon \). Let \( F = \{x \in X : h(x) \geq h(p) + \epsilon\} \). So \( p \notin F \) and \( F \) is closed because \( h \) is continuous. Thus, by Lemma 5, there is \( f \in L \) such that \( f(p) = h(p) + \epsilon, f \geq h(p) + \epsilon \) and \( f > M \) and thus \( f > h \) on \( F \). Notice that on \( \overline{F} \), \( h < h(p) + \epsilon < f \). Consequently, \( f > h \). Hence \( f \in l \). Also because \( 0 \leq f - h = \epsilon \leq \epsilon \) and \( \epsilon \) is arbitrary, \( h(p) = \inf_{f \in l} f(p) \), as desired.

Before we prove the Stone-Weierstrass Theorem (Theorem 1), we still have an important polynomial approximation lemma. It builds a connection from algebras to lattices. The sequence in detail in the background of a linear function space is algebra→polynomial→absolute value function→lattice.

**Lemma 6.** Given \( \epsilon > 0 \), there is a polynomial \( p \) such that if \( x \in [-1, 1] \), then we have \( |p(x) - x| < \epsilon \).

**Proof.** Notice that the series of the function \( f(x) = (1 - x)^2 \) expanded at 0 converges uniformly on the compact set \([0, 1]\), i.e., given \( \epsilon > 0 \), there is \( N \) such that if \( n \geq N \), \(|(1 - x)^2 - f_n(x)| < \epsilon \) for all \( x \in [0, 1] \) where \( f_n \) is the partial sum of the first \( n + 1 \) terms in the series expansion. Let \( p_N(x) = f_N(1 - x^2) \).

So \( p_N \) satisfies \(|x| - p_N(x)| < \epsilon \) for all \( x \in [-1, 1] \).

Finally, we prove the Stone-Weierstrass theorem.

**Proof.** We want to show \( \overline{A} = C(X) \). First, given \( A \) as an algebra, we want to show \( \overline{A} \) is an algebra. Then we show that \( \overline{A} \) is a lattice. Hence \( \overline{A} = \overline{\overline{A}} = C(X) \) by Proposition 3.

First of all, let \( a, b \in \mathbb{R}, f, g \in \overline{A} \) and let \( < f_n >, < g_n > \) be in \( A \) such that \( f_n \to f \) and \( g_n \to g \) (uniformly). It is easy to see that \( af_n + bg_n \to af + bg \). Hence, \( af + bg \in \overline{A} \) which shows \( \overline{A} \) is a linear space. Furthermore, since \( f_n g_n \to fg \), we know \( \overline{A} \) is an algebra.

Let \( f = 0 \), then \( |f| = 0 \in \overline{A} \). If not, \( ||f|| \neq 0 \). Consider \( g = \frac{f}{||f||} \). So \( g \in \overline{A} \). By Lemma 6, for any \( \epsilon > 0 \), there is a polynomial \( p \) such that \( ||g(x)| - p(g(x))|| < \epsilon \) for all \( x \in X \). Notice that \( p \circ g \in \overline{A} \) because \( \overline{A} \) is an algebra. So \( |g| \in \overline{A} = \overline{\overline{A}} = \overline{A} \). Thus, \( |f| \in \overline{A} \).

Notice that for any \( f, g \in \overline{A} \), \( f \lor g = \frac{1}{2}(f + g) + \frac{1}{2}|f + g| \) and \( f \land g = \frac{1}{2}(f + g) - \frac{1}{2}|f + g| \). Thus, \( \overline{A} \) is a lattice, as desired.

**Remark:** To see \( f_n g_n \to fg \), use the inequality \( ||f_n g_n - fg|| \leq ||f_n - f|| \cdot ||g_n|| + ||f|| \cdot ||g_n - g|| \) and notice that there is an \( M \) such that \( ||g_n|| < M \) because of \( ||g_n|| \leq ||g_n - g|| + ||g|| \) and the convergence of \( <g_n> \).