Solution of Exam I

1. Which of the following is the solution of the initial value problem

$$\frac{dy}{dx} = \frac{2x(y-1)}{x^2+3}, \qquad y(0) = 4?$$

Seperable D.E.

A. $y = 3x + 4$

B. $y = \ln(x+1) + 4$

C. $\boxed{y = x^2 + 4}$

D. $x^2 + y^2 = 16$

E. $xy + 4 = y$

$$\frac{dy}{y-1} = \frac{2x}{x^2+3}\,dx = \frac{d(x^2+3)}{x^2+3}$$

$$\ln|y-1| = \ln(x^2+3) + C_1$$

$$\Rightarrow \left.\begin{array}{l} y-1 = c(x^2+3) \\ y(0) = 4 \end{array}\right\} \Rightarrow c = 1$$

$$\Rightarrow y-1 = x^2+3$$

$$\Rightarrow y = x^2+4$$

2. If y is the solution to the equation

$$xy' - 3y = x^3, \qquad y(1) = 1,$$

then $y(e) =$

1st order linear D.E.

A. e^{-3}

B. $2e^{-3}$

C. 0

D. e^3

E. $\boxed{2e^3}$

$$y' - \frac{3}{x}y = x^2$$

$$I(x) = e^{\int -\frac{3}{x}dx} = x^{-3} = \frac{1}{x^3}$$

$$\frac{d}{dx}\left(\frac{1}{x^3}y\right) = \frac{1}{x^3} * x^2 = \frac{1}{x}$$

$$\frac{1}{x^3}y = \ln|x| + c$$

$$\Rightarrow \left.\begin{array}{l} y = x^3(\ln|x| + c) \\ y(1) = 1 \end{array}\right\} \Rightarrow c = 1$$

$$\Rightarrow y = x^3(\ln|x| + 1)$$

$$\Rightarrow y(e) = e^3(\ln e + 1) = 2e^3$$

3. Which of the following is the implicit solution to the initial value problem

$$\underbrace{(y^2 + \cos x)}_{M} + \underbrace{(2xy + \sin y)}_{N}y' = 0, \qquad y\left(\frac{\pi}{2}\right) = 0?$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = 2y \Rightarrow \text{exact D.E.}$$

A. $xy^2 + \cos x + \sin y = 0$

B. $xy^2 + \sin x + \cos y = 2$

C. $xy^2 - \sin x - \cos y = -2$

D. $\boxed{xy^2 + \sin x - \cos y = 0}$

E. $xy^2 - \sin x + \cos y = 0$

$$\int (y^2 + \cos x)\,dx = xy^2 + \sin x$$

$$\int (2xy + \sin y)\,dy = xy^2 - \cos y$$

$$\Rightarrow \phi(x, y) = xy^2 + \sin x - \cos y$$

$$\left.\begin{array}{l} \text{general solution: } xy^2 + \sin x - \cos y = C \\ y(\frac{\pi}{2}) = 0 \end{array}\right\}$$

$$\Rightarrow C = \frac{\pi}{2} \times 0^2 + \sin\frac{\pi}{2} - \cos 0 = 0$$

$$\Rightarrow xy^2 + \sin x - \cos y = 0$$

4. The general solution to the differential equation

$$\underbrace{y' = \frac{x}{2y} + \frac{y}{x}}, \qquad x > 0$$

1st-order homogeneous D.E.

is

A. $y^2 = x(\ln x + C)$

B. $\boxed{y^2 = x^2(\ln x + C)}$

C. $y^2 = x\ln x + C$

D. $y^2 = x^2\ln x + C$

E. $y = x^2\ln x + C$

$$v = \frac{y}{x}$$

$$\left.\begin{array}{l} y' = v + x\frac{dv}{dx} \\ \frac{x}{2y} + \frac{y}{x} = \frac{1}{2v} + v \end{array}\right\}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{1}{2v}$$

$$\Rightarrow 2v\,dv = \frac{dx}{x}$$

$$\left.\begin{array}{l} \Rightarrow v^2 = \ln x + C \\ v = \frac{y}{x} \end{array}\right\} \Rightarrow y^2 = x^2(\ln x + C)$$

5. An object with initial temperature 32F is placed in a refrigerator whose temperature is a constant 0F. An hour later the temperature of the object is 16F. What will its temperature be four hours after it is placed in the refrigerator? Hint: Newton's law of cooling $\frac{dT}{dt} = -k(T - T_m)$.

A. $1F$

B. $\boxed{2F}$

C. $3F$

D. $4F$

E. $5F$

$$\left.\begin{array}{l}\frac{dT}{dt} = -k(T-T_m) \\ T_m = 0\end{array}\right\} \Rightarrow \frac{dT}{dt} = -kT \quad \text{— seperable DE}$$

$$\Rightarrow \left.\begin{array}{l}T = Ce^{-kt} \\ T(0) = 32\end{array}\right\} \Rightarrow C = 32 \Rightarrow \left.\begin{array}{l}T = 32e^{-kt} \\ T(1) = 16\end{array}\right\}$$

$$\Rightarrow 16 = 32e^{-k} \Rightarrow \frac{1}{2} = e^{-k} \Rightarrow \ln\frac{1}{2} = -k \Rightarrow k = \ln 2$$

$$\Rightarrow T = 32e^{-t\ln 2} \Rightarrow T(4) = 32e^{-4\ln 2} = 32 \times 2^{-4} = 2$$

OR by checking the numbers (half life)

$$\underset{t=0}{32} \longrightarrow \underset{t=1}{16} \longrightarrow \underset{t=2}{8} \longrightarrow \underset{t=3}{4} \longrightarrow \boxed{\underset{t=4}{2}}$$

6. Initially a 100-gallon tank is **half full** of pure water. A salt solution containing 0.2lb of salt per gallon runs into the tank at a rate of 3 gallons per minute. The well mixed solution runs out of the tank at a rate of 2 gallons per minute. Let $A(t)$ be the amount of salt in the tank at time t. Then $A(t)$ satisfies the differential equation

A. $\frac{dA}{dt} = 0.6 - \frac{3A}{t+50}$

B. $\frac{dA}{dt} = 0.6 - \frac{2A}{50-t}$

C. $\frac{dA}{dt} = 0.6 + \frac{2A}{t+50}$

D. $\boxed{\frac{dA}{dt} = 0.6 - \frac{2A}{t+50}}$

E. $\frac{dA}{dt} = 0.6 - \frac{2A}{t+100}$

$$\left.\begin{array}{l}V_0 = \frac{100}{2} = 50 \text{ (half full)} \\ \frac{dV}{dt} = r_1 - r_2 = 3 - 2 = 1 \Rightarrow V = t + C\end{array}\right\}$$

$$\Rightarrow V = t + 50$$

$$\frac{dA}{dt} = c_1 r_1 - c_2 r_2 = 0.2 \times 3 - \frac{A}{V} \times 2$$

$$= 0.6 - \frac{2A}{t+50}$$

7. Consider the equation

Bernoulli

$$\frac{dy}{dx} - \frac{1}{4x\ln x}y = 2xy^3. \longrightarrow y^{-3}\frac{dy}{dx} - \frac{1}{4x\ln x}y^{-2} = 2x$$

$$V = y^{-2} \Rightarrow \frac{dv}{dx} = -2y^{-3}\frac{dy}{dx}$$

$$\Rightarrow y^{-3}\frac{dy}{dx} = -\frac{1}{2}\frac{dv}{dx}$$

If $v = y^{-2}$, then v satisfies

A. $\boxed{\frac{dv}{dx} + \frac{1}{2x\ln x}v = -4x}$

B. $\frac{dv}{dx} - \frac{1}{2x\ln x}v = -4x$

C. $\frac{dv}{dx} + \frac{1}{4x\ln x}v = -2x$

D. $\frac{dv}{dx} - \frac{1}{4x\ln x}v = -2x$

E. $\frac{dv}{dx} + \frac{1}{2x\ln x}v = -2x$

$$\Rightarrow -\frac{1}{2}\frac{dv}{dx} - \frac{1}{4x\ln x}V = 2x$$

$$\Rightarrow \frac{dV}{dx} + \frac{1}{2x\ln x}V = -4x$$

(multiply both sides by -2)

8. The number of equilibrium solutions to the differential equation

$$\frac{dy}{dx} = y(y-1)(y-2)(y-x)$$

is

$$\frac{dy}{dx} = 0 \Rightarrow \begin{array}{l} y=0 \\ y=1 \\ y=2 \\ y=x \end{array}$$

A. 0

B. 1

C. 2

D. $\boxed{3}$

E. 4

But $y=x$ is not a constant.

Thus the number of equilibrium solutions is 3

9. If $A = \begin{bmatrix} 1 & x \\ y & 2 \end{bmatrix}$, find all the values of x and y for which $AA^T = \begin{bmatrix} 2 & 4 \\ 4 & 8 \end{bmatrix}$.

$\boxed{\text{A. } x = 1, \quad y = 2}$

B. $x = 2, \quad y = 2$

C. $x = 1, \quad y = 1$

D. $x = 2, \quad y = 1$

E. $x = \pm 1, \quad y = \pm 2$

$$A = \begin{bmatrix} 1 & x \\ y & 2 \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} 1 & y \\ x & 2 \end{bmatrix}$$

$$\Rightarrow AA^T = \begin{bmatrix} 1 & x \\ y & 2 \end{bmatrix}\begin{bmatrix} 1 & y \\ x & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1+x^2 & y+2x \\ y+2x & y^2+4 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 4 & 8 \end{bmatrix}$$

$$\Rightarrow \left.\begin{aligned} 1+x^2 &= 2 \Rightarrow x = \pm 1 \\ y+2x &= 4 \quad\quad y+2x=4 \\ y^2+4 &= 8 \Rightarrow y = \pm 2 \end{aligned}\right\} \Rightarrow \boxed{\begin{aligned} x&=1 \\ y&=2 \end{aligned} \text{ only}}$$

10. Find the general solution to

$$y'' + \frac{1}{x}y' = 3x, \quad x > 0.$$

2^{nd} order D.E with y terms missing

let $v = y'$ $\quad y'' = v'$

A. $y = \frac{1}{3}x^3 + c_1 x + c_2$

B. $y = x^3 + \frac{c_1}{x} + c_2$

C. $y = x^2 + \frac{c_1}{x} + c_2$

D. $y = x^3 + c_1 \ln x + c_2$

$\boxed{\text{E. } y = \frac{1}{3}x^3 + c_1 \ln x + c_2}$

$$\Rightarrow v' + \frac{1}{x}v = 3x \quad \text{— } 1^{st} \text{ order linear D.E.}$$

$$I(x) = e^{\int \frac{1}{x}dx} = x$$

$$\frac{d}{dx}(xv) = x * 3x = 3x^2$$

$$\Rightarrow xv = \int 3x^2 dx + C_1 = x^3 + C_1$$

$$\Rightarrow \left.\begin{aligned} v &= x^2 + \frac{C_1}{x} \\ v &= y' \end{aligned}\right\} \Rightarrow \frac{dy}{dx} = x^2 + \frac{C_1}{x}$$

$$\Rightarrow y = \int \left(x^2 + \frac{C_1}{x}\right)dx + C_2$$

$$= \frac{1}{3}x^3 + C_1 \ln x + C_2$$