MATH 262 Quiz 3 January 29

Name: Solution PID: ____________

Solve the problem systematically and neatly and show all your work.

(3pts) 1. A container initially contains 10 L of water in which there is 20 g of salt dissolved. A solution containing 4 g/L of salt is pumped into the container at a rate of 2 L/min, and the well-stirred mixture runs out at a rate of 2 L/min. What is the concentration in the tank after 10 minutes?

Sol: $V_0 = 10$, $A_0 = 20$, $C_1 = 4$, $r_1 = r_2 = 2$

$$\left.\begin{array}{l} \frac{dV}{dt} = r_1 - r_2 = 0 \\ V(0) = V_0 = 10 \end{array}\right\} \Rightarrow V = 10$$

$$\frac{dA}{dt} = C_1 r_1 - C_2 r_2 = 8 - 2\frac{A}{V}$$

$$\Rightarrow \frac{dA}{dt} = 8 - \frac{1}{5}A = -\frac{1}{5}(A-40)$$

$$\frac{dA}{A-40} = -\frac{1}{5}dt$$

$$\ln|A-40| = -\frac{1}{5}t + C_1$$

$$A - 40 = ce^{-\frac{1}{5}t}$$

$$\left.\begin{array}{l} A = 40 + ce^{-\frac{1}{5}t} \\ A(0) = 20 \end{array}\right\}$$

$$\Rightarrow c = -20$$

$$\Rightarrow A = 40 - 20e^{-\frac{1}{5}t}$$

$$\Rightarrow C_2(10) = \frac{A(10)}{V(10)} = \frac{40 - 20e^{-2}}{10} = \boxed{4 - 2e^{-2}}$$

(3pts) 2. Find the general solution of

$$y' + 2x^{-1}y = 6y^2x^4.$$

Sol. divide the D.E. both sides by y^2

$$\frac{1}{y^2}y' + 2x^{-1}\frac{1}{y} = 6x^4 \quad \boxed{\text{(Bernoulli)}}$$

let $V = \frac{1}{y} \Rightarrow \frac{dV}{dx} = -\frac{1}{y^2}\frac{dy}{dx}$

$$\Rightarrow -\frac{dV}{dx} + \frac{2}{x}V = 6x^4$$

$$\Rightarrow \frac{dV}{dx} - \frac{2}{x}V = -6x^4 \quad \boxed{\text{1st order linear}}$$

$$I(x) = e^{-\int \frac{2}{x}dx} = e^{-2\ln x} = \frac{1}{x^2}$$

$$\frac{d}{dx}\left(\frac{1}{x^2}V\right) = \frac{1}{x^2} * (-6x^4) = -6x^2$$

$$\Rightarrow \frac{1}{x^2}V = \int -6x^2\,dx + C$$

$$\left.\begin{array}{l} \Rightarrow \frac{1}{x^2}V = -2x^3 + C \\ V = \frac{1}{y} \end{array}\right\}$$

$$\Rightarrow \boxed{\frac{1}{y} = -2x^5 + Cx^2 \quad \text{or} \quad y = \frac{1}{-2x^5 + Cx^2}}$$

(4pts) 3. Solve the initial value problem

$$(3x^2 + y)dx + (x + 2y)dy = 0, \quad y(1) = 2.$$

Sol: $\left.\begin{array}{l} M = 3x^2 + y \\ N = x + 2y \end{array}\right\} \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = 1 \Rightarrow \boxed{\text{exact D.E.}}$

$\frac{\partial \phi}{\partial x} = M = 3x^2 + y \Rightarrow \phi(x,y) = \int (3x^2 + y)dx + f(y) = x^3 + xy + f(y)$

$\Rightarrow \frac{\partial \phi}{\partial y} = x + f'(y) = N = x + 2y$

$\Rightarrow f'(y) = 2y \Rightarrow f(y) = y^2$ (No arbitrary constant needed!)

$\Rightarrow \phi(x,y) = x^3 + xy + y^2$

$\left.\begin{array}{l} \Rightarrow x^3 + xy + y^2 = C \text{ — general solution to D.E.} \\ y(1) = 2 \end{array}\right\} \Rightarrow C = 7 \Rightarrow \boxed{x^3 + xy + y^2 = 7}$

Bonus problem (3pts) Find the equation of orthogonal trajectories to the family of

$$x^2 + y^2 - 2cx = 0.$$

Sol: $x^2 + y^2 - 2cx = 0$

$\Rightarrow \frac{d}{dx}(x^2 + y^2 - 2cx) = \frac{d}{dx}(0)$

$\Rightarrow 2x + 2y\frac{dy}{dx} - 2c = 0$

$\left.\begin{array}{l} \Rightarrow \frac{dy}{dx} = \frac{c - x}{y} \\ x^2 + y^2 - 2cx = 0 \Rightarrow c = \frac{x^2 + y^2}{2x} \end{array}\right\}$

$\left.\begin{array}{l} \Rightarrow \frac{dy}{dx} = \frac{y^2 - x^2}{2xy} = m_1 \\ m_1 * m_2 = -1 \end{array}\right\}$

$\Rightarrow m_2 = \frac{2xy}{x^2 - y^2}$

Solve $\frac{dy}{dx} = \frac{2xy}{x^2 - y^2}$

$\boxed{\text{1st-order homogeneous D.E.}}$

$\left.\begin{array}{l} \text{let } v = \frac{y}{x} \Rightarrow y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx} \\ \frac{2xy}{x^2 - y^2} = \frac{2v}{1 - v^2} \end{array}\right\}$

$\Rightarrow x\frac{dv}{dx} = \frac{v + v^3}{1 - v^2}$ $\boxed{\text{— (seperable D.E.)}}$

$\int \left(\frac{1 - v^2}{v + v^3}\right) dv = \int \frac{dx}{x}$

$\frac{1 - v^2}{v + v^3} = \frac{1}{v} - \frac{2v}{1 + v^2}$ (partial fraction Decomposition)

$\Rightarrow \ln|v| - \ln|1 + v^2| = \ln|x| + C_1$

$\left.\begin{array}{l} \Rightarrow \frac{v}{1 + v^2} = kx \\ v = \frac{y}{x} \end{array}\right\} \Rightarrow \frac{\frac{y}{x}}{1 + (\frac{y}{x})^2} = kx$

$\Rightarrow \boxed{y = k(x^2 + y^2) \quad \text{OR} \quad x^2 + y^2 = ky}$