MATH 262 Quiz 7

Name: Solution PID: ________ Section: ________

Solve the problem systematically and neatly and show all your work.

1. (2pts) Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be the linear transformation for which

$$T\left(\begin{bmatrix}1\\1\end{bmatrix}\right) = \begin{bmatrix}1\\2\\3\end{bmatrix} \quad \text{and} \quad T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}0\\3\\1\end{bmatrix}, \text{find} \quad T\left(\begin{bmatrix}1\\2\end{bmatrix}\right).$$

Sol: $\begin{bmatrix}1\\2\end{bmatrix} = c_1\begin{bmatrix}1\\1\end{bmatrix} + c_2\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}c_1+c_2\\c_1\end{bmatrix} \Rightarrow \begin{matrix}c_1 = 2\\ c_2 = -1\end{matrix}$

$\Rightarrow T\left(\begin{bmatrix}1\\2\end{bmatrix}\right) = c_1 T\left(\begin{bmatrix}1\\1\end{bmatrix}\right) + c_2 T\left(\begin{bmatrix}1\\0\end{bmatrix}\right)$

$= 2\begin{bmatrix}1\\2\\3\end{bmatrix} - \begin{bmatrix}0\\3\\1\end{bmatrix} = \boxed{\begin{bmatrix}2\\1\\5\end{bmatrix}}$

2. (3pts) Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be the linear transformation given by $T(x) = Ax$ where $A = \begin{bmatrix}1 & 2 & 3 & 1\\ 2 & 4 & 6 & 2\\ -1 & -2 & -3 & -1\end{bmatrix}$, find $ker(T)$ and $Rng(T)$.

Sol: $A \sim^{REF} \begin{bmatrix}1 & 2 & 3 & 1\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$

$A\vec{x} = \vec{0} \Rightarrow x_1 + 2x_2 + 3x_3 + x_4 = 0 \Rightarrow \begin{cases} x_1 = -2r-3s-t\\ x_2 = r\\ x_3 = s\\ x_4 = t\end{cases}$

$\Rightarrow ker(T) = \text{nullspace}(A) = \{\vec{x} \in \mathbb{R}^4 : \vec{x} = (-2r-3s-t, r, s, t)\}$

$= \text{span}\{(-2,1,0,0), (-3,0,1,0), (-1,0,0,1)\}$

$Rng(T) = \text{colspace}(A) = \text{span}\left\{\begin{bmatrix}1\\2\\-1\end{bmatrix}\right\} = \left\{k\begin{bmatrix}1\\2\\-1\end{bmatrix}, k \in \mathbb{R}\right\}$

3. (5pts) Let $A = \begin{bmatrix}1 & 2 & 2\\ -1 & 4 & 1\\ 0 & 0 & 3\end{bmatrix}$.

(a) Find all the eigenvalues of A.

(b) Find all the eigenspaces of A.

(c) Determine whether A is defective or nondefective.

Sol: (a) $p(\lambda) = \det(A-\lambda I) = \begin{vmatrix} 1-\lambda & 2 & 2 \\ -1 & 4-\lambda & 1 \\ 0 & 0 & 3-\lambda \end{vmatrix}$

expand along 3rd row $= (3-\lambda)\begin{vmatrix} 1-\lambda & 2 \\ -1 & 4-\lambda \end{vmatrix}$

$= (3-\lambda)(\lambda^2-5\lambda+6)$

$= -(\lambda-2)(\lambda-3)^2 = 0$

$\Rightarrow$ eigenvalues are $\boxed{\lambda_1 = 2,\ \lambda_2 = \lambda_3 = 3}$

(b)

(i) for $\lambda_1 = 2$, solve $(A-\lambda_1 I)\vec{v} = \vec{0} \Rightarrow \begin{bmatrix} -1 & 2 & 2 \\ -1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow \left.\begin{array}{l} v_1 = 2v_2 \\ v_3 = 0 \end{array}\right\} \Rightarrow \vec{v} = (2r, r, 0)$

may choose $\vec{v}_1 = (2,1,0) \Rightarrow \boxed{E_1 = \text{span}\{(2,1,0)\}}$

(ii) for $\lambda_2 = \lambda_3 = 3$ solve $(A-\lambda_2 I)\vec{v} = \vec{0} \Rightarrow \begin{bmatrix} -2 & 2 & 2 \\ -1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} -2 & 2 & 2 & 0 \\ -1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} \boxed{1} & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\Rightarrow \left.\begin{array}{l} v_1 = v_2 + v_3 \\ v_2 = r \\ v_3 = s \end{array}\right\} \Rightarrow \left.\begin{array}{l} v_1 = r+s \\ v_2 = r \\ v_3 = s \end{array}\right\} \Rightarrow \vec{v} = (r+s, r, s)$

$= (r,r,0) + (s,0,s)$

$= r(1,1,0) + s(1,0,1)$

there are 2 L.I. vectors $\vec{v}_2 = (1,1,0)$, $\vec{v}_3 = (1,0,1)$ corresponding to eigenvalue $\lambda = 3$

$\Rightarrow \boxed{\begin{array}{l} E_2 = \{\vec{v} \in \mathbb{R}^3 : \vec{v} = r(1,1,0) + s(1,0,1), r, s \in \mathbb{R}\} \\ = \text{span}\{(1,1,0), (1,0,1)\} \end{array}}$

(c) A is $\boxed{\text{nondefective}}$ because it has $n = 3$ L.I eigenvectors $\vec{v}_1 = (2,1,0)$, $\vec{v}_2 = (1,1,0)$, $\vec{v}_3 = (1,0,1)$