Solutions to Exam 2,

If you have a different version, please check the problem number to the left.

1. Determine all the values of a for which the system has infinitely many solutions.

$$
\begin{aligned}
x_1 + 2x_2 - x_3 &= 3 \\
3x_1 + 7x_2 + 5x_3 &= 9 \\
x_1 + x_2 - a^2x_3 &= -a
\end{aligned}
$$

#2

A. $a = 0$ only.

B. $a = 3$ only.

$\boxed{\text{C.}}$ $a = -3$ only.

D. $a = -3$ or $a = 3$.

E. $a = -3$ and $a = 3$.

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 3 & 7 & 5 & 9 \\ 1 & 1 & -a^2 & -a \end{array}\right] \sim \overset{\text{REF}}{\left[\begin{array}{ccc|c} \boxed{1} & 2 & -1 & 3 \\ 0 & \boxed{1} & 8 & 0 \\ 0 & 0 & 9-a^2 & -3-a \end{array}\right]}$$

it has ∞ many solutions if $r = r^\# < n$

$$\Rightarrow \left.\begin{aligned} 9 - a^2 &= 0 \\ -3 - a &= 0 \end{aligned}\right\} \Rightarrow a = -3$$

2. Let $\mathbf{A}$ be an $n \times n$ nonsingular matrix. Which of the following statements must be true?

#1

(i) $det\mathbf{A} = 0$.

✓(ii) $rank(\mathbf{A}) = n$.

(iii) $\mathbf{Ax} = \mathbf{0}$ has infinitely many solutions.

✓(iv) $\mathbf{Ax} = \mathbf{b}$ has a unique solution for every n vector $\mathbf{b}$.

✓(v) $\mathbf{A}$ must be row equivalent to the identity matrix $\mathbf{I_n}$.

A. (i) and (iii) only.

B. (ii) and (iv) only.

C. (i), (iv) and (v).

D. (i), (iii) and (v).

$\boxed{\text{E.}}$ (ii), (iv) and (v).

A is $n \times n$ nonsingular matrix

$\Rightarrow \det(A) \neq 0 \Rightarrow A^{-1}$ exists

$\Rightarrow A\vec{x} = \vec{0}$ has unique sol. $\vec{x} = \vec{0}$

$A\vec{x} = \vec{b}$ has unique sol $\vec{x} = A^{-1}\vec{b}$

$\Rightarrow A \sim I_n$

$\Rightarrow \text{rank}(A) = n$

3. Which of the following sets of vectors forms a basis for $\mathbb{R}^3$? $n = \dim[\mathbb{R}^3] = 3$

$\cancel{\text{A.}}$ $\left\{ \begin{bmatrix} 1 \\ 0 \\ -5 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$ $k = 2 < n = 3$

B. $\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} \right\}$ $\left| \begin{bmatrix} 1 & 0 & 1 \\ 2 & 0 & 3 \\ 1 & 0 & 5 \end{bmatrix} \right| = 0$

C. $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 4 \end{bmatrix} \right\}$ $\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 1 & 2 & 4 \end{vmatrix} = 1 \times \begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} = 0$

#4

✓ D. $\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} \right\}$ $\begin{vmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 2 & 2 & 3 \end{vmatrix} = -1 \times \begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} = -1 \neq 0$

$\cancel{\text{E.}}$ $\left\{ \begin{bmatrix} 1 \\ 0 \\ -5 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 6 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} \right\}$ $k = 4 > n = 3$

4. For what value of k is the vector $(2, 2, 1, 1)$ in the span of $(1, 2, 1, -1)$ and $(3, 2, 1, k)$?

A. -1

B. 0

#3

C. 1

D. 2

$\boxed{\text{E}}$ 3

$$c_1(1, 2, 1, -1) + c_2(3, 2, 1, k) = (2, 2, 1, 1)$$

$$\Rightarrow \left.\begin{aligned} c_1 + 3c_2 &= 2 \\ 2c_1 + 2c_2 &= 2 \\ c_1 + c_2 &= 1 \end{aligned}\right\} \Rightarrow \begin{aligned} c_1 &= 1/2 \\ c_2 &= 1/2 \end{aligned}$$

$$-c_1 + k c_2 = 1 \Rightarrow -\tfrac{1}{2} + \tfrac{1}{2}k = 1$$

$$\Rightarrow k = 3$$

5. Which of the following subset S is a subspace of V?

(i) $V = \mathbb{R}^3$ and S is the set of vectors (x, y, z) satisfying $x + 2y - 3z = 0$.

(ii) $V = M_2(\mathbb{R})$ and S is the set of 2×2 matrices with determinant $\neq 0$.

(iii) $V = P_2$(the set of polynomials of degree at most 2) and S is the set of polynomials of the form $ax^2 - bx$.

(iv) $V = M_n(\mathbb{R})$ and S is the set of $n \times n$ nonsymmetric matrices.

#6

A. (i) and (iii) only.

B. (i) and (iv) only.

C. (ii) and (iii) only.

D. (i) (iii) and (iv).

E. (i) (ii) and (iii).

(i) $S = \{ \vec{v} = (-2y+3z, y, z) \}$

$\vec{v_1} + \vec{v_2} = (-2(y_1+y_2)+3(z_1+z_2), y_1+y_2, z_1+z_2) \in S$ ✓

$k\vec{v_1} = (-2ky_1+3kz_1, ky_1, kz_1) \in S$ ✓

(ii) $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \in S$ But $A+B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \notin S$

(iii) $S = \{ p(x) = ax^2 - bx,\ a, b \in \mathbb{R} \}$

$p_1(x) + p_2(x) = (a_1+a_2)x^2 - (b_1+b_2)x \in S$ ✓

$kp_1(x) = kax^2 - kbx \in S$ ✓

(iv) $A = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$ $B = \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix} \in S$

But $A + B = \underbrace{\begin{bmatrix} 4 & 3 \\ 3 & 6 \end{bmatrix}}_{\text{symmetric matrix}} \notin S$

6. Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 0 & 0 & 1 \end{bmatrix}$ and $B = A^{-1}$, then the entry b_{23} of A^{-1} is

#5

A. $\frac{3}{2}$

B. $-\frac{3}{2}$

C. 3

D. -3

E. 0

$$b_{23} = \frac{C_{32}}{\det(A)} = \frac{(-1)^{3+2} M_{32}}{\det(A)}$$

$$M_{32} = \begin{vmatrix} 1 & 3 \\ 0 & 3 \end{vmatrix} = 3$$

$$\det(A) = 2$$

$\Rightarrow b_{23} = -\frac{3}{2}$

OR $\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 2 & 3 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 3 \\ 0 & 1 & 0 & 0 & \frac{1}{2} & -3/2 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$

OR $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} * & * & b_{13} \\ * & * & b_{23} \\ * & * & b_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \left.\begin{array}{l} 2b_{23} + 3b_{33} = 0 \\ b_{33} = 1 \end{array}\right\} \Rightarrow b_{23} = -\frac{3}{2}$

7. Let

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix},$$

which of the following sets is a basis of the *nullspace*(A)?

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$\boxed{\text{A.}}$ $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

B. $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ -3 \end{bmatrix}$

C. $\begin{bmatrix} 3 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$

D. $\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$

E. $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$

$A\vec{x} = \vec{0}$

$A^{\#} = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 2 & 2 & 2 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} \boxed{1} & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$\Rightarrow \quad x_1 + x_2 + x_3 = 0 \quad \Rightarrow x_1 = -r-s$

$x_2 = r$

$x_3 = s$

$\Rightarrow$ nullspace$(A) = \{\vec{x} = (-r-s, r, s), r, s \in \mathbb{R}\}$

2 free variables r & s

$r=1, s=0 \Rightarrow \vec{x} = (-1, 1, 0)$

$r=0, s=1 \Rightarrow \vec{x} = (-1, 0, 1)$ $\Bigg\} \Rightarrow$ A

8. The Wronskian of the functions $\{x, \cos x, \sin x\}$ is

#7

A. $-x$

B. 0

$\boxed{\text{C.}}$ x

D. $x(\sin^2 x - \cos^2 x)$

E. $x \cos x \sin x$

$W[x, \cos x, \sin x](x) = \begin{vmatrix} x & \cos x & \sin x \\ 1 & -\sin x & \cos x \\ 0 & -\cos x & -\sin x \end{vmatrix}$

$\overset{A_{31}(1)}{=} \begin{vmatrix} x & 0 & 0 \\ 1 & -\sin x & \cos x \\ 0 & -\cos x & -\sin x \end{vmatrix}$

$\overset{\text{along 1st row}}{=} x \begin{vmatrix} -\sin x & \cos x \\ -\cos x & -\sin x \end{vmatrix}$

$= x(\sin^2 x + \cos^2 x)$

$= x$

9. If A and B are $3{\times}3$ matrices such that $\det(A) = 3$ and $\det(B) = 4$, then $\det(-2AB^{-1}) =$

A. -24

$\boxed{\text{B}}$ -6

C. 6

D. $\frac{3}{2}$

E. $-\frac{3}{2}$

#9

$$\det(-2AB^{-1}) = \det(-2A)\det(B^{-1})$$
$$= (-2)^3 \det(A) * \frac{1}{\det(B)}$$
$$= (-8) * 3 * \frac{1}{4}$$
$$= -6$$

10. Let $A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 2 & 1 \\ 2 & 4 & 0 & 2 & -2 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$, which of the following is a basis of the *colspace*(A)?

A. $\left\{ \begin{bmatrix} 1\\1\\2\\0 \end{bmatrix}, \begin{bmatrix} 2\\2\\4\\0 \end{bmatrix}, \begin{bmatrix} 3\\3\\0\\0 \end{bmatrix} \right\}$

B. $\left\{ \begin{bmatrix} 1\\1\\2\\0 \end{bmatrix}, \begin{bmatrix} 2\\2\\4\\0 \end{bmatrix}, \begin{bmatrix} 4\\2\\2\\1 \end{bmatrix} \right\}$

C. $\left\{ \begin{bmatrix} 1\\1\\2\\0 \end{bmatrix}, \begin{bmatrix} 2\\2\\4\\0 \end{bmatrix}, \begin{bmatrix} 5\\1\\-2\\2 \end{bmatrix} \right\}$

$\boxed{\text{D.}}$ $\left\{ \begin{bmatrix} 1\\1\\2\\0 \end{bmatrix}, \begin{bmatrix} 3\\3\\0\\0 \end{bmatrix}, \begin{bmatrix} 4\\2\\2\\1 \end{bmatrix} \right\}$

E. $\left\{ \begin{bmatrix} 1\\1\\2\\0 \end{bmatrix}, \begin{bmatrix} 4\\2\\2\\1 \end{bmatrix}, \begin{bmatrix} 5\\1\\-2\\2 \end{bmatrix} \right\}$

#8

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 2 & 1 \\ 2 & 4 & 0 & 2 & -2 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} \boxed{1} & 2 & 3 & 4 & 5 \\ 0 & 0 & \boxed{1} & 1 & 2 \\ 0 & 0 & 0 & \boxed{1} & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Choose the 1st, 3^{rd} & 4^{th} columns in A to be a basis for colspace(A)