Solution to Spring 2001 Final Exam

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1. Seperable D.E.

$$\frac{dy}{\sqrt{y}} = (4x+2)\,dx$$

$$\left.\begin{array}{l} \Rightarrow 2y^{\frac{1}{2}} = 2x^2+2x+C \\ y(1)=1 \end{array}\right\} \Rightarrow C=-2$$

$$\Rightarrow y^{\frac{1}{2}} = x^2+x-1 \Rightarrow y=(x^2+x-1)^2$$

$$\Rightarrow y(2) = (2^2+2-1)^2 = \boxed{25} \Rightarrow C$$

2. homogeneous 1^{st}-order D.E.

$$\left.\begin{array}{l} \text{Let } v=\frac{y}{x} \Rightarrow y=vx \Rightarrow \frac{dy}{dx} = x\frac{dv}{dx}+v \\ \frac{x^2}{3y^2}+\frac{y}{x} = \frac{1}{3v^2}+v \end{array}\right\}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{1}{3v^2} \Rightarrow 3v^2\,dv = \frac{dx}{x}$$

$$\Rightarrow v^3 = \ln|x| + C, \quad v=\frac{y}{x}$$

$$\left.\begin{array}{l} \Rightarrow y^3 = x^3(\ln x + c) \\ y(1)=2 \end{array}\right\} \Rightarrow c=8$$

$$\Rightarrow \boxed{y^3 = x^3(\ln x+8) = x^3\ln x + 8x^3} \Rightarrow E$$

3. Exact D.E.

$$\left.\begin{array}{l} M=2xy \\ N=x^2+1 \end{array}\right\} \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = 2x$$

$$\frac{\partial \phi}{\partial x} = M = 2xy \Rightarrow \phi = x^2y + h(y)$$

$$\Rightarrow \frac{\partial \phi}{\partial y} = x^2 + h'(y) = N$$

$$\Rightarrow h'(y)=1 \Rightarrow h(y)=y$$

$$\Rightarrow \phi = x^2y+y$$

$$\Rightarrow \boxed{x^2y+y=C} \Rightarrow D$$

4. $\frac{dT}{dt} = -k(T-80) \Rightarrow \frac{dT}{T-80} = -k\,dt$

$$\Rightarrow T-80 = ce^{-kt} \Rightarrow T = 80+ce^{-kt}$$

$$\left.\begin{array}{l} T(2)=320 \Rightarrow 320 = 80+ce^{-2k} \\ T(3)=240 \Rightarrow 240 = 80+ce^{-3k} \end{array}\right\}$$

$$\Rightarrow e^k = \frac{240}{160} = \frac{3}{2} \Rightarrow k = \ln\frac{3}{2}$$

$$C = 240 * e^{2k} = 240*\left(\frac{3}{2}\right)^2 = 540$$

$$\Rightarrow T = 80+540e^{-\ln\frac{3}{2}t} \Rightarrow \boxed{T(0)=620F}$$

B

5. $\left.\begin{array}{l} \frac{dV}{dt} = 2-2=0 \Rightarrow V=c \\ V_0 = 4 \end{array}\right\} \Rightarrow V=4$

$$\frac{dA}{dt} = 2\times 1 - 2*\frac{A}{V} = 2-\frac{A}{2} = -\frac{1}{2}(A-4)$$

$$\left.\begin{array}{l} \Rightarrow \frac{dA}{A-4} = -\frac{1}{2}dt \Rightarrow A = 4+ce^{-\frac{1}{2}t} \\ A(0)=8 \end{array}\right\}$$

$$\Rightarrow C=4 \Rightarrow A = 4+4e^{-\frac{1}{2}t}$$

D

$$\Rightarrow \text{Concentration } C_2 = \frac{A}{V} = \frac{4+4e^{-\frac{1}{2}t}}{4} = \boxed{1+e^{-\frac{1}{2}t}}$$

6. $M \sim^{REF} \begin{bmatrix} \boxed{1} & 4 & -11 \\ 0 & \boxed{1} & -3 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \boxed{\text{rank}=2} \Rightarrow C$

7. $M^{-1} = \dfrac{\begin{bmatrix} 2 & -1 \\ -2 & 3 \end{bmatrix}}{\begin{vmatrix} 3 & 1 \\ 2 & 2 \end{vmatrix}} = \frac{1}{4}\begin{bmatrix} 2 & -1 \\ -2 & 3 \end{bmatrix} \Rightarrow B$

8. $\det(2A^{-1}B^2) = 2^3\det(A^{-1})(\det(B))^2 = \frac{8}{2}\times 3^2$

$$= \boxed{36} \quad A$$

9. $A^{-1} = \frac{1}{\det(A)}\,\text{adj}(A) = \frac{1}{\det(A)}\begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$

$$C_{ij} = (-1)^{i+j}M_{ji}$$

$$\left.\begin{array}{l} \Rightarrow b_{23} = \frac{1}{\det(A)}(-1)^{2+3}M_{32} \\ M_{32} = \begin{vmatrix} 1 & 3 & 4 \\ 0 & 5 & 7 \\ 0 & 1 & 2 \end{vmatrix} = 3 \quad \det(A)=10 \end{array}\right\} \Rightarrow \boxed{b_{23} = -\frac{3}{10}}$$

$$\Rightarrow B$$

10. $A \sim \begin{bmatrix} \boxed{1} & 0 & 2 & 9 \\ 0 & \boxed{1} & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\dim[\mathrm{Rng}(T)] = \dim[\text{colspace}(A)] = \text{rank}(A) = 2 \Rightarrow$ (B)

11. (i) $y_1 \in S,\ y_2 \in S \Rightarrow y_1 + y_2 \notin S$ b/c

$(y_1+y_2)'' + (y_1+y_2)' - 5(y_1+y_2) = 2 \neq 1$

(ii) $k(0,1,0) = (0,k,0) \in U$ if $k \neq 0, 1$

(iii) $V = \{(x, -x), x \in \mathbb{R}\}$ is closed under $\oplus$ and $\odot$ $\Rightarrow$ (D)

12. only need to check A, B, E by determinant

determinant of E $= 12 \neq 0 \Rightarrow$ (E)

13. $\begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix} = c_1 \begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix} + c_2 \begin{bmatrix} 5 & 3 \\ 2 & a \end{bmatrix}$

$$\Rightarrow \left.\begin{array}{l} \left.\begin{array}{l} 2c_1 + 5c_2 = 1 \\ c_1 + 3c_2 = 1 \\ 2c_2 = 2 \end{array}\right\} \Rightarrow c_1 = -2 \quad c_2 = 1 \\ -c_1 + ac_2 = 0 \end{array}\right\}$$

$\Rightarrow \boxed{a = -2} \Rightarrow$ (D)

14. Let $(1,0) = c_1(1,2) + c_2(3,2)$

$\Rightarrow c_1 = -\frac{1}{2} \quad c_2 = \frac{1}{2}$ (A)

$\Rightarrow T(1,0) = -\frac{1}{2}T(1,2) + \frac{1}{2}T(3,2) = \boxed{(-2,1)}$

5. $AA^T = \begin{bmatrix} 1 & y \\ x & -1 \end{bmatrix} \begin{bmatrix} 1 & x \\ y & -1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 5 \end{bmatrix}$

$$\Rightarrow \left.\begin{array}{l} 1 + y^2 = 2 \\ x - y = 1 \\ x^2 + 1 = 5 \end{array}\right\} \Rightarrow x = 2,\ y = 1 \Rightarrow \text{(B)}$$

6. Annihilator to

$$\left.\begin{array}{lcl} \cos x & - & D^2+1 \\ xe^x & - & (D-1)^2 \\ 1 & - & D \end{array}\right\}$$

$\Rightarrow$ (D)

17. $r = -1$ is one of the solutions of $r^2 - 3r - 4 = 0$

$$\left.\begin{array}{l} \Rightarrow \text{trial for } xe^{-x} \text{ is } x(Ax+B)e^{-x} \\ \quad\ \text{trial for } \cos 2x \text{ is } C\cos 2x + D\sin 2x \end{array}\right\}$$

$\Rightarrow$ (D)

18. $r^2 + r - 6 = 0 \Rightarrow r_1 = 2 \quad r_2 = -3$

$\Rightarrow y_1 = e^{2x} \quad y_2 = e^{-3x}$

$W = W[y_1, y_2] = \begin{vmatrix} e^{2x} & e^{-3x} \\ 2e^{2x} & -3e^{-3x} \end{vmatrix} = -5e^{-x}$

$u_2' = \dfrac{y_1 F}{W} = \dfrac{e^{2x} * -5\ln(1+x^2)}{-5e^{-x}} =$ (A)

19. $y = ue^{-x} \Rightarrow y' = -ue^{-x} + u'e^{-x}$

$y'' = ue^{-x} - 2u'e^{-x} + u''e^{-x}$

plug all into the D.E. (ignore u terms)

$x(-2u'e^{-x} + u''e^{-x}) + (1+2x)u'e^{-x} = 0$

$\Rightarrow xe^{-x}u'' + e^{-x}u' = 0 \Rightarrow \boxed{xu'' + u' = 0}$

(D)

20. $(y_1 - y_2)'' + a_1(y_1 - y_2)' + a_2(y_1 - y_2) = 0$

$(y_1 + y_2)'' + a_1(y_1 + y_2)' + a_2(y_1 + y_2) = 2F(x)$

$(2y_1 - y_2)'' + a_1(2y_1 - y_2)' + a_2(2y_1 - y_2) = F(x)$

$(y_1 - 2y_2)'' + a_1(y_1 - 2y_2)' + a_2(y_1 - 2y_2) = -F(x)$

$\Rightarrow$ (C)

21. $r^2 + 3r - 4 = 0 \Rightarrow r_1 = 1 \quad r_2 = -4$

$\Rightarrow y_c(x) = c_1 e^x + c_2 e^{-4x}$

$y_p(x) = Ae^{2x} \Rightarrow y_p' = 2Ae^{2x}$

$y_p'' = 4Ae^{2x}$

$\Rightarrow 4Ae^{2x} + 6Ae^{2x} - 4Ae^{2x} = 6e^{2x} \Rightarrow A = 1$

$$\left.\begin{array}{l} \Rightarrow y(x) = c_1 e^x + c_2 e^{-4x} + e^{2x} \\ y(0) = 2 \qquad y'(0) = 3 \end{array}\right\} \Rightarrow \begin{array}{l} c_1 = 1 \\ c_2 = 0 \end{array}$$

$\Rightarrow y(x) = e^x + e^{2x} \Rightarrow y(1) = e + e^2 \Rightarrow$ (D)

22. $A - \lambda I = \begin{bmatrix} 1-\lambda & 2 & 2 \\ -1 & 4-\lambda & 1 \\ 0 & 0 & 3-\lambda \end{bmatrix}$

$\lambda = 3$ solve $\begin{bmatrix} -2 & 2 & 2 \\ -1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \vec{v} = \vec{0}$

$B^{\#} \sim \begin{bmatrix} 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\Rightarrow \left.\begin{matrix} v_1 - v_2 - v_3 = 0 \\ v_2 = r \\ v_3 = s \end{matrix}\right\} \Rightarrow \vec{v} = \begin{bmatrix} r+s \\ r \\ s \end{bmatrix}$

$\Rightarrow \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

checking the given answers, the correct one is C b/c $\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \vec{v}_1 + \vec{v}_2 \Rightarrow$ C

23. $A - \lambda I = \begin{bmatrix} 1-\lambda & 2 \\ 0 & 3-\lambda \end{bmatrix} \Rightarrow \lambda_1 = 1 \quad \lambda_2 = 3$ distinct

$\Rightarrow$ there are 2 L.I. eigenvectors

$B - \lambda I = \begin{bmatrix} 1-\lambda & 2 \\ 0 & 1-\lambda \end{bmatrix} \Rightarrow \lambda_1 = \lambda_2 = 1$

Solve $\begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} \vec{v} = \vec{0} \Rightarrow \vec{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

only 1 eigenvector corresponding to $\lambda_1 = \lambda_2 = 1$

$\Rightarrow$ B is defective

$C - \lambda I = \begin{bmatrix} 1-\lambda & 1 \\ 0 & -\lambda \end{bmatrix} \Rightarrow \lambda_1 = 1 \quad \lambda_2 = 0$

Distinct eigenvalues $\Rightarrow$ 2 L.I eigenvectors

$\Rightarrow$ C is Nondefective

Thus the answer is C

24. Example in Section 7.4.

$A = \begin{bmatrix} 3 & 0 & 0 \\ 1 & 1 & 2 \\ -1 & 2 & 1 \end{bmatrix}$

$\Rightarrow A - \lambda I = \begin{bmatrix} 3-\lambda & 0 & 0 \\ 1 & 1-\lambda & 2 \\ -1 & 2 & 1-\lambda \end{bmatrix}$

$\Rightarrow \det(A - \lambda I) = (3-\lambda) \begin{vmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix}$

$= (3-\lambda)(\lambda^2 - 2\lambda - 3)$

$= -(\lambda-3)^2(\lambda+1)$

$\Rightarrow \lambda_1 = \lambda_2 = 3 \quad \lambda_3 = -1$

$\lambda = 3$ solve $\begin{bmatrix} 0 & 0 & 0 \\ 1 & -2 & 2 \\ -1 & 2 & -2 \end{bmatrix} \vec{v} = \vec{0}$

$\Rightarrow \vec{v} = \begin{bmatrix} 2r-2s \\ r \\ s \end{bmatrix} \Rightarrow \vec{v}_1 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \quad \vec{v}_2 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$

$\lambda = -1$ solve $\begin{bmatrix} 4 & 0 & 0 \\ 1 & 2 & 2 \\ -1 & 2 & 2 \end{bmatrix} \vec{v} = \vec{0} \Rightarrow \vec{v}_3 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$

$\Rightarrow$ Solution is E

25. $X(t)\vec{u}' = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$,

$\Rightarrow \vec{u}' = [X(t)]^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \dfrac{\begin{bmatrix} 2e^{2t} & -e^{2t} \\ 0 & e^{-2t} \end{bmatrix}}{\begin{vmatrix} e^{-2t} & e^{2t} \\ 0 & 2e^{2t} \end{vmatrix}} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$= \begin{bmatrix} e^{2t} \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} u_1' = e^{2t} \\ u_2' = 0 \end{matrix}$

$\Rightarrow \begin{matrix} u_1 = \int e^{2t}\,dt = \frac{1}{2}e^{2t} \\ u_2 = \int 0\,dt = 0 \end{matrix}$

$\Rightarrow \vec{x}_p(t) = X(t) \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$

$= \begin{bmatrix} e^{-2t} & e^{2t} \\ 0 & 2e^{2t} \end{bmatrix} \begin{bmatrix} \frac{1}{2}e^{2t} \\ 0 \end{bmatrix}$

$=$ B