Solutions to Spring 2002 Final Exam

A C D E B / D C A B A / E E C C B / A E D E C / D A A B C /

1. Exact D.E.

$$\left.\begin{array}{l} M = 2xy + \cos y \\ N = x^2 - x\sin y - 2 \end{array}\right\} \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = 2x - \sin y$$

$$\frac{\partial \phi}{\partial x} = M = 2xy + \cos y \Rightarrow \phi = x^2 y + x\cos y + h(y)$$

$$\Rightarrow \frac{\partial \phi}{\partial y} = x^2 - x\sin y + h'(y) = N$$

$$\Rightarrow h'(y) = -2 \Rightarrow h(y) = -2y$$

$$\Rightarrow \phi = x^2 y + x\cos y - 2y \Rightarrow \text{(A)}$$

2. 1^{st}-order linear $\quad y' + \frac{1}{x}y = \frac{1}{x}e^{5x}$

$$I(x) = e^{\int \frac{1}{x}dx} = x$$

$$\Rightarrow \frac{d}{dx}(xy) = x * \frac{1}{x}e^{5x} = e^{5x}$$

$$\Rightarrow xy = \int e^{5x}dx = \frac{1}{5}e^{5x} + c$$

$$\Rightarrow y = \frac{1}{5x}e^{5x} + \frac{c}{x} \Rightarrow \text{(C)}$$

3. $$\left.\begin{array}{l} v = \frac{y}{x} \Rightarrow y = vx \Rightarrow \frac{dy}{dx} = xv' + v \\ \sin(\frac{y}{x}) = \sin v \end{array}\right\}$$

$$\Rightarrow xv' + v = \sin v \Rightarrow \text{(D)}$$

4. $r^2 + 1 = 0 \Rightarrow r = \pm i \Rightarrow x_c(t) = c_1\cos t + c_2 \sin t$

$$\left.\begin{array}{l} x_p = (At+B)e^t \Rightarrow x_p' = (At+A+B)e^t \\ \qquad x_p'' = (At+2A+B)e^t \end{array}\right\}$$

plug them into the D.E.

$$\Rightarrow (2At + 2A + 2B)e^t = 2te^t$$

$$\Rightarrow A = 1 \quad B = -1 \Rightarrow x_p = (t-1)e^t$$

$$\left.\begin{array}{l} \Rightarrow x(t) = c_1\cos t + c_2\sin t + (t-1)e^t \\ x(0) = -1 \qquad x'(0) = 0 \end{array}\right\}$$

$$\Rightarrow c_1 = c_2 = 0$$

$$\Rightarrow x(t) = (t-1)e^t \Rightarrow x(1) = 0 \Rightarrow \text{(E)}$$

5. $V_0 = 20 \quad A_0 = 20 \times 5 = 100$

$c_1 = 10 \quad r_1 = r_2 = 2$

$$\left.\begin{array}{l} \frac{dV}{dt} = r_1 - r_2 = 0 \\ V(0) = V_0 = 20 \end{array}\right\} \Rightarrow V = 20$$

$$\frac{dA}{dt} = c_1 r_1 - c_2 r_2 = 20 - \frac{A}{20} * 2 = -\frac{1}{10}(A - 200)$$

$$\left.\begin{array}{l} \Rightarrow A = 200 + Ce^{-\frac{1}{10}t} \\ A(0) = A_0 = 100 \end{array}\right\} \Rightarrow C = -100$$

$$\Rightarrow A = 200 - 100e^{-\frac{1}{10}t} \Rightarrow A(10) = \boxed{200 - 100e^{-1}}$$

(B)

6. $$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \\ 3 & 2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \text{(D)}$$

7. $$A^{\#} = \begin{bmatrix} 1 & 0 & 2 & -1 & 0 \\ 2 & -1 & 5 & -2 & 0 \\ 0 & 1 & -1 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 2 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{array}{l} x_1 = -2r + s \\ x_2 = r \\ x_3 = r \\ x_4 = s \end{array} \Rightarrow \vec{x} = \begin{bmatrix} -2r+s \\ r \\ r \\ s \end{bmatrix}$$

$$\Rightarrow \text{basis is (C)} \quad \left\{\begin{array}{l} r = 1 \quad s = 0 \\ r = 0 \quad s = 1 \end{array}\right.$$

8. $(1,2,3) = 1(1,0,0) + 2(0,1,0) + 3(0,0,1)$

$$\Rightarrow T(1,2,3) = T(1,0,0) + 2T(0,1,0) + 3T(0,0,1)$$

$$= (0,0,-1) + (2,2,0) + (3,3,6)$$

$$= (5,5,5) \Rightarrow \text{(A)}$$

9. $\begin{vmatrix} 2 & -k & 0 \\ 1 & 2 & 2 \\ 0 & 1 & -k \end{vmatrix} = \begin{vmatrix} 0 & -k-4 & -4 \\ 1 & 2 & 2 \\ 0 & 1 & -k \end{vmatrix}$

$= - \begin{vmatrix} -k-4 & -4 \\ 1 & -k \end{vmatrix} = -(k^2+4k+4) = 0$

$\Rightarrow k = -2 \Rightarrow$ (B)

10. $(2, -k, 0) = c(1, 2, 2) \Rightarrow$ (A)

11. (E) b/c there are 4 vectors in $\mathbb{R}^3$.

12. $A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A) = \frac{1}{\det(A)} \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$

$C_{ij} = (-1)^{i+j} M_{ji}$

$$\left.\begin{array}{l} \Rightarrow b_{23} = \frac{1}{\det(A)} C_{23} = \frac{1}{\det(A)} (-1)^5 M_{32} \\ \det(A) = 1 \times 2 \times 3 = 6 \\ M_{32} = \begin{vmatrix} 1 & 3 \\ 0 & 4 \end{vmatrix} = 4 \end{array}\right\}$$

$\Rightarrow b_{23} = (-1)\frac{4}{6} = -\frac{2}{3} \Rightarrow$ (E)

13. $A^{\#} = \begin{bmatrix} 1 & 1 & 3 & 1 \\ 2 & 4 & 5 & 1 \\ 1 & -1 & k^2 & -k \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 1 & 3 & 1 \\ 0 & 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & k^2-4 & -k-2 \end{bmatrix}$

$r \neq r^{\#} \Rightarrow \left.\begin{array}{l} k^2-4=0 \\ -k-2 \neq 0 \end{array}\right\} \Rightarrow k = 2 \Rightarrow$ (C)

14. $r^2 + r - 2 = 0 \Rightarrow r_1 = 1 \quad r_2 = -2$

$$\left.\begin{array}{l} \Rightarrow y_c = c_1 e^{x} + c_2 e^{-2x} \\ y_p = \ln x \end{array}\right\}$$

$\Rightarrow y = c_1 e^{x} + c_2 e^{-2x} + \ln x \Rightarrow$ (C)

15. $r^2 - r - 2 = 0 \Rightarrow r_1 = -1 \quad r_2 = 2$

$\left.\begin{array}{l} \Rightarrow y = c_1 e^{-x} + c_2 e^{2x} \\ y(0) = 1 \quad y'(0) = -1 \end{array}\right\} \Rightarrow \begin{array}{l} c_1 = 1 \\ c_2 = 0 \end{array}$

$\Rightarrow y = e^{-x} \Rightarrow y(1) = e^{-1} \Rightarrow$ (B)

16. $y = x^r \Rightarrow y' = r x^{r-1}, \; y'' = r(r-1)x^{r-2}$

plug all into the D.E.

$\Rightarrow r(r-1)x^r + 5r x^r + 4x^r = 0$

$\Rightarrow r^2 + 4r + 4 = 0 \Rightarrow r = -2 \Rightarrow$ (A)

17. $r^4 + 2r^2 + 1 = 0 \Rightarrow r = \pm i$ of $m = 2$

$\Rightarrow$ (E)

18. $r^3 - 4r = 0 \Rightarrow r_1 = 0, \; r_2 = 2 \quad r_3 = -2$

$\Rightarrow$ trial solution for $3\cos t \to A\cos t + B\sin t$

$e^{-2t} \to c t e^{-2t}$

$\Rightarrow$ (D)

19. $F(x) = x^{-1}e^{-x}$ Need to use variation of parameter method.

$r^2 + 2r + 1 = 0 \Rightarrow r_1 = r_2 = -1$

$\Rightarrow y_1 = e^{-x}, \; y_2 = x e^{-x}$

$\Rightarrow W[y_1, y_2] = \begin{vmatrix} e^{-x} & x e^{-x} \\ -e^{-x} & (1-x)e^{-x} \end{vmatrix} = e^{-2x}$

$\Rightarrow u_1' = -\frac{y_2 F}{W} = -\frac{x e^{-x} * x^{-1} e^{-x}}{e^{-2x}} = -1$

$u_2' = \frac{y_1 F}{W} = \frac{e^{-x} * x^{-1} e^{-x}}{e^{-2x}} = \frac{1}{x}$

$\Rightarrow u_1 = -x \qquad u_2 = \ln x$

$\Rightarrow y_p(x) = -x e^{-x} + \ln x \cdot x e^{-x}$

By checking the given answers, we may chose $u_2 = \ln x + 1$ Then $y_p(x) =$ (E)

20. The graph is for underdamping

(2 complex conjugate roots for $r^2+\alpha r+1=0$)

$$r^2+\alpha r+1=0 \Rightarrow r=\frac{-\alpha\pm\sqrt{\alpha^2-4}}{2}$$

$$\Rightarrow \left.\begin{matrix}\alpha^2-4<0\\ -\frac{\alpha}{2}<0\end{matrix}\right\} \Rightarrow 0<\alpha<2 \Rightarrow \text{(C)}$$

21.
$$A-\lambda I=\begin{bmatrix}2-\lambda & 0 & 0\\ 1 & 3-\lambda & 4\\ 0 & -1 & -1-\lambda\end{bmatrix}$$

$$\Rightarrow \det(A-\lambda I)=(2-\lambda)\begin{vmatrix}3-\lambda & 4\\ -1 & -1-\lambda\end{vmatrix}$$
$$=(2-\lambda)(\lambda^2-2\lambda+1)$$
$$=(2-\lambda)(\lambda-1)^2=0 \Rightarrow \text{(D)}$$

22.
$$A=\begin{bmatrix}1 & 2\\ 4 & 3\end{bmatrix}$$

$$A-\lambda I=\begin{bmatrix}1-\lambda & 2\\ 4 & 3-\lambda\end{bmatrix}$$

$$\det(A-\lambda I)=\lambda^2-4\lambda-5=0 \Rightarrow \lambda_1=-1,\ \lambda_2=5$$

$\lambda_1=-1$ solve $\begin{bmatrix}2 & 2\\ 4 & 4\end{bmatrix}\vec{v}=\vec{0} \Rightarrow \vec{v}_1=\begin{bmatrix}1\\ -1\end{bmatrix}$

$\lambda_2=5$ solve $\begin{bmatrix}-4 & 2\\ 4 & -2\end{bmatrix}\vec{v}=\vec{0} \Rightarrow \vec{v}_2=\begin{bmatrix}1\\ 2\end{bmatrix}$

$$\Rightarrow \vec{x}(t)=c_1e^{\lambda_1t}\vec{v}_1+c_2e^{\lambda_2t}\vec{v}_2 = \text{(A)}$$

23.
$$A=\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}$$

$$A-\lambda I=\begin{bmatrix}-\lambda & 1\\ -1 & -\lambda\end{bmatrix}$$

$$\det(A-\lambda I)=0 \Rightarrow \lambda^2+1=0 \Rightarrow \lambda=\pm i$$

$\lambda_1=i$ solve $\begin{bmatrix}-i & 1\\ -1 & -i\end{bmatrix}\vec{v}=\vec{0} \Rightarrow \vec{v}_1=\begin{bmatrix}1\\ i\end{bmatrix}$

$$\Rightarrow \vec{w}_1(t)=e^{\lambda_1t}\vec{v}_1=e^{it}\begin{bmatrix}1\\ i\end{bmatrix}$$
$$=\begin{bmatrix}\cos t+i\sin t\\ -\sin t+i\cos t\end{bmatrix}=\begin{bmatrix}\cos t\\ -\sin t\end{bmatrix}+i\begin{bmatrix}\sin t\\ \cos t\end{bmatrix}$$

$$\left.\begin{matrix}\Rightarrow \vec{x}(t)=c_1\begin{bmatrix}\cos t\\ -\sin t\end{bmatrix}+c_2\begin{bmatrix}\sin t\\ \cos t\end{bmatrix}\\ x_1(0)=1\quad x_2(0)=1 \Rightarrow \vec{x}(0)=\begin{bmatrix}1\\ 1\end{bmatrix}\end{matrix}\right\}$$

$$\Rightarrow c_1=c_2=1 \Rightarrow \vec{x}(t)=\begin{bmatrix}\cos t+\sin t\\ -\sin t+\cos t\end{bmatrix}$$

$$\Rightarrow x_2(t)=-\sin t+\cos t \Rightarrow \text{(A)}$$

24.
$$\vec{w}_1(t)=e^{rt}\vec{v}=e^{(2+i)t}\begin{bmatrix}1\\ -1-i\end{bmatrix}$$
$$=e^{-2t}(\cos t+i\sin t)\begin{bmatrix}1\\ -1-i\end{bmatrix}$$
$$=e^{-2t}\begin{bmatrix}\cos t+i\sin t\\ -\cos t+\sin t+i(-\cos t-\sin t)\end{bmatrix}$$

$$\left.\begin{matrix}\Rightarrow \vec{x}_c(t)=c_1e^{-2t}\begin{bmatrix}\cos t\\ -\cos t+\sin t\end{bmatrix}+\\ c_2e^{-2t}\begin{bmatrix}\sin t\\ -\cos t-\sin t\end{bmatrix}\\ \vec{x}_p(t)=\begin{bmatrix}1\\ 2\end{bmatrix}e^{-2t}\end{matrix}\right\}$$

$$\Rightarrow \vec{x}(t)=\text{(B)}$$

25. (A) $A\vec{x}=\vec{b}$ has $\begin{cases}0 \text{ solution if } \vec{b}\neq\vec{0}\\ \infty \text{ solutions if } \vec{b}=\vec{0}\end{cases}$

(B) $A\vec{x}=\vec{0}$ has ∞ solutions

(C) ✓ b/c it has ∞ solutions

(D) if $A=\begin{bmatrix}2 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 0\end{bmatrix}$ $\lambda=1$ is NOT an eigenvalue of A

(E) $\det(A)=0 \Rightarrow A^{-1}$ doesn't exist

Thus the answer is (C)