1. If y is the solution to the equation

$$y' = \frac{x^2+1}{y^3}, \qquad y(0) = 2,$$

then $y(3) =$

seperable D.E.

A. ± 2
B. $\pm 2\sqrt{2}$
C. $\pm 4\sqrt{2}$
D. 0
E. ± 4

$y^3\,dy = (x^2+1)\,dx$

$\left.\begin{array}{l} \frac{1}{4}y^4 = \frac{1}{3}x^3 + x + C \\ y(0) = 2 \end{array}\right\}$

$\Rightarrow \frac{1}{4} * 2^4 = 0 + 0 + C \Rightarrow C = 4$

$\Rightarrow y^4 = \frac{4}{3}x^3 + 4x + 16$

$\Rightarrow y^4(3) = \frac{4}{3} \times 3^3 + 4 \times 3 + 16 = 64$

$\Rightarrow y(3) = \pm\sqrt[4]{64} = \boxed{\pm 2\sqrt{2}}$

2. If y is the solution to the equation

$$y' = \frac{3}{x}y + x, \qquad y(1) = 0,$$

then $y(\sqrt{3}) =$

1st order linear $\quad \frac{dy}{dx} - \frac{3}{x}y = x$

A. $-3\sqrt{3}$
B. $3\sqrt{3}$
C. $3(\sqrt{3}-1)$
D. $3(\sqrt{3}+1)$
E. $\frac{27-\sqrt{3}}{45}$

$I(x) = e^{-\int \frac{3}{x}dx} = e^{-3\ln x} = \frac{1}{x^3}$

$\frac{d}{dx}\left(\frac{1}{x^3}y\right) = \frac{1}{x^3} * x = \frac{1}{x^2}$

$\Rightarrow \frac{1}{x^3}y = -\frac{1}{x} + c$

$\Rightarrow \left.\begin{array}{l} y = -x^2 + cx^3 \\ y(1) = 0 \end{array}\right\} \Rightarrow c = 1$

$\Rightarrow y = x^3 - x^2 \Rightarrow y(\sqrt{3}) = \boxed{3(\sqrt{3}-1)}$

3. Which of the following is the implicit solution to the initial value problem

$$(e^x \sin y - 2y\sin x - 1) + (e^x \cos y + 2\cos x + 3)y' = 0, \qquad y(0) = \pi?$$

$M = e^x \sin y - 2y \sin x - 1$

$N = e^x \cos y + 2\cos x + 3$

A. $e^x \sin y + 2y\cos x + 3x - y = \pi$

B. $e^x \sin y + 2y\cos x + 3x - y = -1 - \pi$

C. $e^x \sin y + 2y\cos x + 3y - x = \pi$

D. $e^x \sin y + 2y\cos x + 3y - x = 5\pi$

E. $e^x \sin y - 2y\cos x + 3y - x = 5\pi$

$\Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = e^x \cos y - 2\sin x$

$\Rightarrow$ exact D.E.

$\frac{\partial \phi}{\partial x} = M \Rightarrow \phi(x,y) = \int (e^x \sin y - 2y\sin x - 1)\,dx + f(y) = e^x \sin y + 2y\cos x - x + f(y)$

$\Rightarrow \frac{\partial \phi}{\partial y} = e^x \cos y + 2\cos x + f'(y) = N \Rightarrow f'(y) = 3 \Rightarrow f(y) = 3y$

$\Rightarrow \phi(x,y) = e^x \sin y + 2y\cos x - x + 3y$

general soluti $e^x \sin y + 2y\cos x - x + 3y = C$

$y(0) = \pi$

$\Rightarrow C = \pi$

$\Rightarrow \boxed{e^x \sin y + 2y\cos x - x + 3y = \pi}$

4. The general solution of

$$\frac{dy}{dx} = \frac{2xy + 3x^2}{y^2 - x^2}$$

is

1st order homogeneous

A. $\frac{y^3}{3} - x^2 y - x^3 = c$

B. $\frac{y^3}{3} + x^2 y + x^3 = c$

C. $\frac{y^3}{3} - x^2 y + x^3 = c$

D. $2xy + y^2 + 3x^2 = c$

E. $2xy + y^2 - 3x^2 = c$

let $v = \frac{y}{x}$

$\Rightarrow v + x\frac{dv}{dx} = \frac{2v+3}{v^2-1}$

$\Rightarrow x\frac{dv}{dx} = \frac{3v+3-v^3}{v^2-1}$ — seperable D.E.

$\Rightarrow \frac{(v^2-1)\,dv}{3v+3-v^3} = \frac{dx}{x}$

let $w = 3v+3-v^3 \Rightarrow dw = (3-3v^2)\,dv \Rightarrow (v^2-1)dv = -\frac{1}{3}dw$

$\Rightarrow \frac{-\frac{1}{3}dw}{w} = \frac{dx}{x} \Rightarrow \frac{dw}{w} = -3\frac{dx}{x} \Rightarrow \ln|w| = -3\ln|x| + C_1$

$\Rightarrow \ln|w| + 3\ln|x| = C_1 \Rightarrow wx^3 = C$

$w = 3v+3-v^3 = 3\frac{y}{x} + 3 - (\frac{y}{x})^3$

$\Rightarrow 3yx^2 + 3x^3 - y^3 = C \Rightarrow \frac{1}{3}y^3 - x^2 y - x^3 = \underbrace{-\frac{1}{3}C}_{\text{replaced by } c} \Rightarrow$ A

5. Consider the equation

$$\frac{dy}{dx} + \frac{1}{4x}y = 2xy^2.$$

Bernoulli

If $v = y^{-1}$, then v satisfies

A. $\frac{dv}{dx} + \frac{1}{4x}v = -2x$

B. $\frac{dv}{dx} - \frac{1}{4x}v = -2x$

C. $\frac{dv}{dx} + \frac{1}{4x}v = 2x$

D. $\frac{dv}{dx} - \frac{1}{4x}v = 2x$

E. $\frac{dv}{dx} + \frac{1}{2x}v = -2x$

Divide both sides by y^2

$$y^{-2}\frac{dy}{dx} + \frac{1}{4x}y^{-1} = 2X$$

let $V = y^{-1} \Rightarrow \frac{dV}{dX} = -y^{-2}\frac{dy}{dX}$

$$\Rightarrow -\frac{dV}{dx} + \frac{1}{4x}V = 2X$$

$$\Rightarrow \boxed{\frac{dV}{dX} - \frac{1}{4X}V = -2X}$$

6. Two minutes after a hot metal object is taken from a furnace, the temperature of the object is $320F$. One minutes later its temperature is $240F$. If the temperature of the room is 80F, what was the temperature of the object when it was removed from the furnace? Hint: Newton's law of cooing $\frac{dT}{dt} = -k(T - T_m)$.

A. $540F$

B. $620F$

C. $640F$

D. $\frac{1280}{3}F$

E. $\frac{5120}{9}F$

$$\frac{dT}{dt} = -K(T-80) \Rightarrow T = 80 + ce^{-kt}$$

$$T(2) = 320 \Rightarrow 80 + ce^{-2k} = 320$$

$$T(3) = 240 \Rightarrow 80 + ce^{-3k} = 240$$

$$\Rightarrow \left.\begin{array}{l} ce^{-2k} = 240 \\ ce^{-3k} = 160 \end{array}\right\} \Rightarrow e^{k} = \frac{240}{160} = \frac{3}{2}$$

$$\left.\begin{array}{l} \Rightarrow k = \ln\frac{3}{2} \\ ce^{-2k} = 240 \end{array}\right\} \Rightarrow C = 540$$

$$\Rightarrow T = 80 + 540e^{-kt}$$

$$\Rightarrow T(0) = 80 + 540 = \boxed{620F}$$

7. Initially a tank holds 100-gallon of pure water. A salt solution containing 0.2lb of salt per gallon runs into the tank at a rate of 3 gallons per minute. The well mixed solution runs out of the tank at a rate of 2 gallons per minute. Let $A(t)$ be the amount of salt in the tank at time t. Then $A(t)$ satisfies the differential equation

A. $\frac{dA}{dt} = 0.2 - \dfrac{3A}{t+100}$

B. $\frac{dA}{dt} = 0.4 - \dfrac{3A}{t+100}$

C. $\frac{dA}{dt} = 0.6 - \dfrac{3A}{t+100}$

D. $\frac{dA}{dt} = 0.4 - \dfrac{2A}{t+100}$

E. $\frac{dA}{dt} = 0.6 - \dfrac{2A}{t+100}$

$V_0 = 100 \quad A_0 = 0$

$C_1 = 0.2 \quad r_1 = 3 \quad r_2 = 2 \quad C_2 = \frac{A}{V}$

$\frac{dV}{dt} = r_1 - r_2 = 3 - 2 = 1 \Rightarrow \left.\begin{array}{l} V = t + C \\ V(0) = 100 \end{array}\right\}$

$\Rightarrow C = 100$

$\Rightarrow V = t + 100$

$\frac{dA}{dt} = C_1 r_1 - C_2 r_2 = 0.6 - \frac{A}{t+100} * 2$

$= \boxed{0.6 - \frac{2A}{t+100}}$

8. The equilibrium solution(s) to the differential equation

$$\frac{dy}{dx} = y(y-1)(y-2)$$

are

A. $y = 0, 1$

B. $y = 0, 2$

C. $y = 1, 2$

D. $y = 0, 1, 2$

E. None of the above.

Let $y(y-1)(y-2) = 0$

$\Rightarrow y = 0, \ y = 1, \ y = 2$

all are constants $\Rightarrow$ D

9. For what values of a, b and c is it true that

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}?$$

Sol.

$$\left.\begin{aligned} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} &= \begin{bmatrix} a & b \\ 0 & 2c \end{bmatrix} \\ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} &= \begin{bmatrix} a & 2b \\ 0 & 2c \end{bmatrix} \end{aligned}\right\} \Rightarrow \begin{cases} a = a \\ b = 2b \Rightarrow b = 0 \\ 0 = 0 \\ 2c = 2c \end{cases}$$

$\Rightarrow$ $\boxed{\text{a, c can be any number} \quad b = 0}$

10. Find the general solution to

$$(x-1)(x-2)y'' = y' - 1, \quad x > 2.$$

Sol: 2nd order D.E. of case I $\quad y'' = F(x, y') = \dfrac{y'-1}{(x-1)(x-2)}$

let $v = y'$ $\quad \dfrac{dv}{dx} = y''$

$\Rightarrow \dfrac{dv}{dx} = \dfrac{v-1}{(x-1)(x-2)}$ — separable D.E.

(i) $v = 1$ — equilibrium

(ii) $v \neq 1$ $\quad \dfrac{1}{v-1} dv = \dfrac{1}{(x-1)(x-2)} dx = \left(\dfrac{1}{x-2} - \dfrac{1}{x-1}\right) dx$ partial fraction decomposition

$\Rightarrow \ln|v-1| = \ln\left|\dfrac{x-2}{x-1}\right| + C \Rightarrow v = 1 + C_1\left(\dfrac{x-2}{x-1}\right) = 1 + C_1\left(1 - \dfrac{1}{x-1}\right)$

$= (1 + C_1) - \dfrac{C_1}{x-1} \quad C_1 \neq 0$

Combine (i) & (ii)

$$\left.\begin{aligned} v &= (1+C_1) - \frac{C_1}{x-1} \quad C_1 \text{ - any number} \\ v &= \frac{dy}{dx} \end{aligned}\right\}$$

$\Rightarrow y = \int\left[(1+C_1) - \dfrac{C_1}{x-1}\right]dx + C_2 = \boxed{(1+C_1)x - C_1\ln(x-1) + C_2}$