Solution to Practice Exam 2

1. $A^{\#} = \begin{bmatrix} 1 & 1 & 1 & 2 \\ 2 & 3 & 2 & 5 \\ 1 & 3 & a^2-3 & a+2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & a^2-4 & a-2 \end{bmatrix}$

The system has no solution if $r \neq r^{\#} \Rightarrow \left.\begin{matrix} a^2-4=0 \\ a-2\neq 0 \end{matrix}\right\} \Rightarrow \boxed{a=-2 \text{ only}}$

2. use Wronskian to check first, if $W=0$, then use definition to check.

for A $\quad W = 2 \neq 0 \Rightarrow$ L.I.

3. $\dim[P_2] = 3$ = # of vectors in the given set, So only need to check L.I.

use Wronskian

$$W = \begin{vmatrix} 1+kx^2 & 1+x+x^2 & 2+x \\ 2kx & 1+2x & 1 \\ 2k & 2 & 0 \end{vmatrix} = -2(1+k) \neq 0 \Rightarrow \boxed{k \neq -1}$$

4. $AA^{-1} = \begin{bmatrix} 1 & 2 & 1 \\ -1 & 4 & 1 \\ 2 & -4 & 0 \end{bmatrix} \begin{bmatrix} x & b_{12} & x \\ x & b_{22} & x \\ x & b_{32} & x \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$\Rightarrow \left.\begin{aligned} b_{12} + 2b_{22} + b_{32} &= 0 \\ -b_{12} + 4b_{22} + b_{32} &= 1 \\ 2b_{12} - 4b_{22} &= 0 \end{aligned}\right\} \Rightarrow \left\{\begin{aligned} b_{12} &= -1 \\ b_{22} &= -\tfrac{1}{2} \\ \boxed{b_{32} = 2} \end{aligned}\right.$$

5. verify (i) and (ii) by definition — closure under (+) and (·)

(iii) $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ are nonsingular But $A+B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ singular

(iv) $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ are singular But $A+B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ nonsingular

6. $\begin{bmatrix} 5 & -2 \\ 2 & a \end{bmatrix} = c_1 \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} + c_2 \begin{bmatrix} -1 & 2 \\ 1 & 0 \end{bmatrix} + c_3 \begin{bmatrix} 2 & 0 \\ 3 & 2 \end{bmatrix}$

$$\Rightarrow \left.\begin{aligned} c_1 - c_2 + 2c_3 &= 5 \\ 2c_2 &= -2 \\ c_2 + 3c_3 &= 2 \\ a &= 2c_1 + 2c_3 \end{aligned}\right\} \Rightarrow \left.\begin{aligned} c_1 &= 2 \\ c_2 &= -1 \\ c_3 &= 1 \end{aligned}\right\} \Rightarrow a = 6$$

7. An example for this question is $Ax = 0$ $\boxed{\begin{matrix} x_1 + x_2 = 0 \\ 2x_1 + 3x_2 = 0 \\ 4x_1 + 5x_2 = 0 \end{matrix}}$ $\Rightarrow$ C

8. $\det(A(-2B)^{-1}) = \dfrac{\det(A)}{\det(-2B)} = \dfrac{-3}{(-2)^4 \times 2} = \boxed{-\dfrac{3}{32}}$

9. $\text{span}\{\vec{v_1}, \vec{v_2}, \vec{v_3}, \vec{v_4}\}$ = rowspace of A where $A = \begin{bmatrix} 1 & -1 & 1 & -1 \\ 3 & 1 & 7 & 3 \\ 5 & -3 & 9 & 1 \\ -2 & 4 & 2 & 8 \end{bmatrix}$

$$A \sim \begin{bmatrix} \boxed{1} & -1 & 1 & -1 \\ 0 & \boxed{1} & 2 & 3 \\ 0 & 0 & \boxed{1} & 5/4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \dim[\text{rowspace}(A)] = 3$$

10. the determinant of B is Nonzero

11. $r = 1, s = 0 \Rightarrow \vec{x_1} = (1, 1, 2)$
$r = 0 \ s = 1 \Rightarrow \vec{x_2} = (1, -1, 2)$ $\Big\} \Rightarrow$ A

12. $A \sim^{REF} \begin{bmatrix} \boxed{1} & 1 & -1 & 1 \\ 0 & \boxed{1} & 2 & -3 \\ 0 & 0 & 0 & \boxed{1} \end{bmatrix}$

(a)
$\Rightarrow$ rowspace$(A) = \{(1, 1, -1, 1), (0, 1, 2, -3), (0, 0, 0, 1)\}$

(b)
colspace$(A) = \{(1, 1, 3), (1, 0, 2), (1, 4, 2)\}$

(c)
$A\vec{x} = \vec{0} \Rightarrow \begin{matrix} x_1 + x_2 - x_3 + x_4 = 0 \\ x_2 + 2x_3 - 3x_4 = 0 \\ x_4 = 0 \end{matrix} \Rightarrow \begin{matrix} x_1 = 3r \\ x_2 = -2r \\ x_3 = r \\ x_4 = 0 \end{matrix}$

$\Rightarrow \vec{x} = (3r, -2r, r, 0)$
$r = 1$ $\Big\} \Rightarrow$ $\vec{x} = (3, -2, 1, 0)$ — basis for nullspace(A)

(d) rank$(A) = 3 \rightarrow$ # of leading 1s or # of nonzero rows

$\dim[\text{nullspace}(A)] = 1$