Supplementary Problems

A For what value(s), if any, of \( A \) will \( y = Axe^{-x} \) be a solution of the differential equation \( 2y' + 2y = e^{-x} \)? For what value(s), if any, of \( B \) will \( y = Be^{-x} \) be a solution?

B Using the substitution \( u(x) = y + x \), solve the differential equation \( \frac{dy}{dx} = (y + x)^2 \).

C Using the substitution \( u(x) = y^3 \), solve the differential equation \( y^2 \frac{dy}{dx} + \frac{y^3}{x} = \frac{1}{x} \) \((x > 0)\).

D Use \texttt{dfield6} to plot the slope field for the differential equation \( y' = 2y - 3e^{-t} \).

Plot the solution satisfying \( y(0) = 1.001 \). What happens to the solution as \( t \to \infty \)?

Plot the solution satisfying \( y(0) = 0.999 \). What happens to this solution as \( t \to \infty \)?

E Find the explicit solution of the Separable Equation \( \frac{dy}{dt} = 4y - y^2 \), \( y(0) = 8 \). What is the largest open interval containing \( t = 0 \) for which the solution is defined?

F The graph of \( f(y) \) vs \( y \) is as shown:

(a) Find the equilibrium solutions of the autonomous differential equation \( \frac{dy}{dt} = f(y) \).
(b) Determine the stability of each equilibrium solution.

G Solve the differential equation \( \frac{d\theta}{dr} = \frac{2r\theta}{\theta^2 - r^2} \).

H (a) If \( y' = -2y + e^{-t} \), \( y(0) = 1 \) then compute \( y(1) \).

(b) Experiment using the Euler Method (\texttt{eul}) with step sizes of the form \( h = \frac{1}{n} \) to find the smallest value of \( n \) which will give a value \( y_n \) that approximates the above true solution at \( t = 1 \) within 0.05.

(I) (a) If \( y' = 2y - 3e^{-t} \), \( y(0) = 1 \) then compute \( y(1) \).

(b) Experiment using the Euler Method (\texttt{eul}) with step sizes of the form \( h = \frac{1}{n} \) to find the smallest value of \( n \) which will give a value \( y_n \) that approximates the above true solution at \( t = 1 \) within 0.05.
Consider the initial value problem \( \begin{cases} y' = 2ty - y^2 \\ y(1) = 2.5 \end{cases} \). Using the Euler, Improved Euler and Runge-Kutta Methods (eul, rk2, rk4 respectively) with \( h = 0.1 \) to complete this table:

<table>
<thead>
<tr>
<th>( t_n )</th>
<th>Euler ( y_n )</th>
<th>Improved Euler ( y_n )</th>
<th>Runge-Kutta ( y_n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.2</td>
<td></td>
<td></td>
<td></td>
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<tr>
<td>1.3</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.4</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.5</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.6</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Choosing smaller and smaller step sizes \( h \) does not guarantee better and better approximations even for a simple initial value problem like \( \begin{cases} \frac{dy}{dt} = t (y - 1) \\ y(-10) = 0 \end{cases} \).

(a) Verify that \( y(t) = 1 - e^{\frac{(t^2 - 100)}{2}} \) is a solution of the above initial value problem.
(b) Approximate the actual solution at \( t = 10 \) (note that \( y(10) = 0 \)) using the Runge-Kutta Method (rk4) with \( h = 0.2, h = 0.1 \) and \( h = 0.05 \) and fill in the table:

<table>
<thead>
<tr>
<th>( h )</th>
<th>Runge-Kutta Approximation at ( t = 10 )</th>
<th>Actual Solution at ( t = 10 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.20</td>
<td></td>
<td>0.0000</td>
</tr>
<tr>
<td>0.10</td>
<td></td>
<td>0.0000</td>
</tr>
<tr>
<td>0.05</td>
<td></td>
<td>0.0000</td>
</tr>
</tbody>
</table>

Approximation methods for differential equations can be used to estimate definite integrals:

(a) Show that \( y(x) = \int_0^x e^{-t^2} \, dt \) satisfies the initial value problem \( \frac{dy}{dx} = e^{-x^2}, \quad y(0) = 0 \).
(b) Use the Runge-Kutta Method (rk4) with \( h = 0.1 \) to approximate \( y(1.5) \), i.e., \( \int_0^{1.5} e^{-t^2} \, dt \).
To transform any 2\textsuperscript{nd} order linear differential equation \( P(t) y'' + Q(t) y' + R(t) y = G(t) \) into an equivalent 1\textsuperscript{st} order linear system of equations

\[
\begin{align*}
    x'_1(t) &= a_{11}(t) x_1(t) + a_{12}(t) x_2(t) + g_1(t) \\
    x'_2(t) &= a_{21}(t) x_1(t) + a_{22}(t) x_2(t) + g_2(t)
\end{align*}
\]

one can use the substitution \( x_1(t) = y \) and \( x_2(t) = y' \). Transform the initial value problem

\[
2y'' + 3y' - ty = 3e^t, \quad y(0) = 1, \quad y'(0) = -4
\]

into an equivalent system of 1\textsuperscript{st} order equations with initial conditions.

If \( y' = xy^2 - y^3 \) and \( y(1) = 2 \), find \( y''(1) \) and \( y'''(1) \).

From the theory of elasticity, if the ends of a horizontal beam (of uniform cross-section and constant density) are supported at the same height in vertical walls, then its vertical displacement \( y(x) \) satisfies the \textit{boundary value problem} \( \begin{cases} y''' = -P \\ y(0) = y(L) = 0 \\ y'(0) = y'(L) = 0 \end{cases} \), where \( P > 0 \) is a constant depending on the beam’s density and rigidity and \( L \) is the distance between supporting walls:

(a) Solve the above boundary value problem when \( L = 4 \) and \( P = 24 \).

(b) Show that the maximum displacement occurs at the center of the beam \( x = \frac{4}{2} = 2 \).

Using Laplace Transforms, solve this \textit{boundary value problem}:

\[
\begin{cases}
y'' + 4y = 16t \\
y(0) = 0 \\
y\left(\frac{\pi}{4}\right) = 0
\end{cases}
\]

\textit{Hint} : Solve the initial value problem \( \begin{cases} y'' + 4y = 16t \\ y(0) = 0 \end{cases} \) and then determine \( A \) from (*)

You can use Laplace transforms to find particular solutions to some nonhomogeneous differential equations. Use Laplace transforms to find a particular solution, \( y_p(t) \), of \( y'' + 4y = 10e^t \).

\textit{Hint} : Solve the initial value problem \( \begin{cases} y'' + 4y = 10e^t \\ y(0) = 0 \\
y'(0) = 0 \end{cases} \).

(You will get a different particular solution if you use Undetermined Coefficients or Variation of Parameters.)
Tank # 1 initially contains 50 gals of water with 10 oz of salt in it, while Tank # 2 initially contains 20 gals of water with 15 oz of salt in it. Water containing 2 oz/gal of salt flows into Tank # 1 at a rate of 5 gal/min and the well-stirred mixture flows from Tank # 1 into Tank # 2 at the same rate of 5 gal/min. The solution in Tank # 2 flows out to the ground at a rate of 5 gal/min. If $x_1(t)$ and $x_2(t)$ represent the number of ounces of salt in Tank # 1 and Tank # 2, respectively, **Set Up But Do Not Solve** an initial value problem describing this system.

If $\tilde{x}^{(1)}(t)$ and $\tilde{x}^{(2)}(t)$ are linearly independent solutions to the $2 \times 2$ system $\tilde{x}' = A \tilde{x}$, then the matrix $\Phi(t) = \begin{pmatrix} \tilde{x}^{(1)}(t) \\ \tilde{x}^{(2)}(t) \end{pmatrix}$ is called a **Fundamental Matrix** for the system. Find a Fundamental Matrix $\Phi(t)$ of the system $\tilde{x}' = \begin{pmatrix} 4 & -3 \\ 8 & -6 \end{pmatrix} \tilde{x}$.

Find a particular solution $\tilde{x}_p(t)$ of these nonhomogeneous systems:

(a) $\tilde{x}' = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} \tilde{x} + \begin{pmatrix} 5 e^{2t} \\ 3 \end{pmatrix}$

(b) $\tilde{x}' = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} \tilde{x} + \begin{pmatrix} 0 \\ 4 e^t \end{pmatrix}$

(c) $\tilde{x}' = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} \tilde{x} + \begin{pmatrix} 10 \cos t \\ 0 \end{pmatrix}$