23. Show that the set of all solutions to the nonhomogeneous differential equation
\[ y'' + a_1 y' + a_2 y = F(x), \]
where \( F(x) \) is nonzero on an interval \( I \), is not a subspace of \( C^2(I) \).

24. Let \( S_1 \) and \( S_2 \) be subspaces of a vector space \( V \).
Let
\[ S_1 \cup S_2 = \{ v \in V : v \in S_1 \text{ or } v \in S_2 \}, \]
\[ S_1 \cap S_2 = \{ v \in V : v \in S_1 \text{ and } v \in S_2 \}, \]

(a) Show that, in general, \( S_1 \cup S_2 \) is not a subspace of \( V \).
(b) Show that \( S_1 \cap S_2 \) is a subspace of \( V \).
(c) Show that \( S_1 + S_2 \) is a subspace of \( V \).

4.4 Spanning Sets

The only algebraic operations that are defined in a vector space \( V \) are those of addition and scalar multiplication. Consequently, the most general way in which we can combine the vectors \( v_1, v_2, \ldots, v_k \) in \( V \) is
\[ c_1 v_1 + c_2 v_2 + \cdots + c_k v_k, \]
(4.4.1)
where \( c_1, c_2, \ldots, c_k \) are scalars. An expression of the form (4.4.1) is called a linear combination of \( v_1, v_2, \ldots, v_k \). Since \( V \) is closed under addition and scalar multiplication, it follows that the foregoing linear combination is itself a vector in \( V \). One of the questions we wish to answer is whether every vector in a vector space can be obtained by taking linear combinations of a finite set of vectors. The following terminology is used in the case when the answer to this question is affirmative:

**DEFINITION 4.4.1**

If every vector in a vector space \( V \) can be written as a linear combination of \( v_1, v_2, \ldots, v_k \), we say that \( V \) is spanned or generated by \( v_1, v_2, \ldots, v_k \) and call the set of vectors \( \{ v_1, v_2, \ldots, v_k \} \) a spanning set for \( V \). In this case, we also say that \( \{ v_1, v_2, \ldots, v_k \} \) spans \( V \).

This spanning idea was introduced in the preceding section within the framework of differential equations. In addition, we are all used to representing geometric vectors in \( \mathbb{R}^3 \) in terms of their components as (see Section 4.1)
\[ v = ai + bj + ck, \]
where \( i, j, \) and \( k \) denote the unit vectors pointing along the positive \( x-, y-, \) and \( z- \)axes, respectively, of a rectangular Cartesian coordinate system. Using the above terminology, we say that \( v \) has been expressed as a linear combination of the vectors \( i, j, \) and \( k \), and that the vector space of all geometric vectors is spanned by \( i, j, \) and \( k \).

We now consider several examples to illustrate the spanning concept in different vector spaces.

**Example 4.4.2**

Show that \( \mathbb{R}^2 \) is spanned by the vectors
\[ v_1 = (1, 1) \quad \text{and} \quad v_2 = (2, -1). \]
Determine whether the vectors \( \mathbf{v}_1 = (1, 1) \) and \( \mathbf{v}_2 = (2, -1) \) in \( \mathbb{R}^2 \) span \( \mathbb{R}^2 \). We must establish that for every \( \mathbf{v} = (x_1, x_2) \) in \( \mathbb{R}^2 \), there exist constants \( c_1 \) and \( c_2 \) such that

\[
\mathbf{v} = c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2.
\]

That is, in component form,

\[
(x_1, x_2) = c_1(1, 1) + c_2(2, -1).
\]

Equating corresponding components in this equation yields the following linear system:

\[
\begin{align*}
2c_1 + 2c_2 &= x_1, \\
 c_1 - c_2 &= x_2.
\end{align*}
\]

In this system, we view \( x_1 \) and \( x_2 \) as fixed, while the variables we must solve for are \( c_1 \) and \( c_2 \). The determinant of the matrix of coefficients of this system is

\[
\begin{vmatrix}
2 & 2 \\
1 & -1
\end{vmatrix} = -3.
\]

Since this is nonzero regardless of the values of \( x_1 \) and \( x_2 \), the matrix of coefficients is invertible, and hence for all \( (x_1, x_2) \) in \( \mathbb{R}^2 \), the system has a (unique) solution according to Theorem 2.6.4. Thus, Equation (4.4.2) can be satisfied for every vector \( \mathbf{v} \) in \( \mathbb{R}^2 \), so the given vectors do span \( \mathbb{R}^2 \). Indeed, solving the linear system yields

\[
\begin{align*}
c_1 &= \frac{1}{4}(x_1 + x_2), \\
c_2 &= \frac{1}{4}(x_1 - x_2).
\end{align*}
\]

Hence,

\[
(x_1, x_2) = \frac{1}{4}(x_1 + x_2)\mathbf{v}_1 + \frac{1}{4}(x_1 - x_2)\mathbf{v}_2.
\]

Determine whether the vectors \( \mathbf{v}_1 = (1, -1, 4) \), \( \mathbf{v}_2 = (-2, 1, 3) \), and \( \mathbf{v}_3 = (4, -3, 5) \) span \( \mathbb{R}^3 \).

**Example 4.4.3**

Let \( \mathbf{v} = (x_1, x_2, x_3) \) be an arbitrary vector in \( \mathbb{R}^3 \). We must determine whether there are real numbers \( c_1, c_2, c_3 \) such that

\[
\mathbf{v} = c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + c_3 \mathbf{v}_3
\]

or, in component form,

\[
(x_1, x_2, x_3) = c_1 (1, -1, 4) + c_2 (-2, 1, 3) + c_3 (4, -3, 5).
\]

Equating corresponding components on either side of this vector equation yields

\[
\begin{align*}
c_1 - 2c_2 + 4c_3 &= x_1, \\
-2c_1 + c_2 - 3c_3 &= x_2, \\
4c_1 + 3c_2 + 5c_3 &= x_3.
\end{align*}
\]
Reducing the augmented matrix of this system to row-echelon form, we obtain
\[
\begin{pmatrix}
1 & -2 & 4 & x_1 \\
0 & 1 & -1 & -x_1 - x_2 \\
0 & 0 & 7x_1 + 11x_2 + x_3
\end{pmatrix}
\]
It follows that the system is consistent if and only if \(x_1, x_2, x_3\) satisfy
\[
7x_1 + 11x_2 + x_3 = 0.
\] (4.4.4)
Consequently, Equation (4.4.3) holds only for those vectors \(v = (x_1, x_2, x_3)\) in \(\mathbb{R}^3\) whose components satisfy Equation (4.4.4). Hence, \(v_1, v_2,\) and \(v_3\) do not span \(\mathbb{R}^3\).
Geometrically, Equation (4.4.4) is the equation of a plane through the origin in space, and so by taking linear combinations of the given vectors, we can obtain only those vectors which lie on this plane. We leave it as an exercise to verify that indeed the three given vectors lie in the plane with Equation (4.4.4). It is worth noting that this plane forms a subspace \(S\) of \(\mathbb{R}^3\), and that while \(V\) is not spanned by the vectors \(v_1, v_2,\) and \(v_3\), \(S\) is.

The reason that the vectors in the previous example did not span \(\mathbb{R}^3\) was because they were coplanar. In general, any three noncoplanar vectors \(v_1, v_2,\) and \(v_3\) in \(\mathbb{R}^3\) span \(\mathbb{R}^3\), since, as illustrated in Figure 4.4.3, every vector in \(\mathbb{R}^3\) can be written as a linear combination of \(v_1, v_2,\) and \(v_3\). In subsequent sections we will make this same observation from a more algebraic point of view.

\begin{center}
\textbf{Figure 4.4.3:} Any three noncoplanar vectors in \(\mathbb{R}^3\) span \(\mathbb{R}^3\).
\end{center}

Notice in the previous example that the linear combination (4.4.3) can be written as the matrix equation
\[
A\mathbf{c} = \mathbf{v},
\]
where the columns of \(A\) are the given vectors \(v_1, v_2,\) and \(v_3\): \(A = [v_1, v_2, v_3]\). Thus, the question of whether or not the vectors \(v_1, v_2,\) and \(v_3\) span \(\mathbb{R}^3\) can be formulated as follows: Does the system \(A\mathbf{c} = \mathbf{v}\) have a solution \(\mathbf{c}\) for every \(\mathbf{v}\) in \(\mathbb{R}^3\)? If so, then the column vectors of \(A\) span \(\mathbb{R}^3\), and if not, then the column vectors of \(A\) do not span \(\mathbb{R}^3\). This reformulation applies more generally to vectors in \(\mathbb{R}^n\), and we state it here for the record.

\textbf{Theorem 4.4.4}

Let \(v_1, v_2, \ldots, v_k\) be vectors in \(\mathbb{R}^n\). Then \(\{v_1, v_2, \ldots, v_k\}\) spans \(\mathbb{R}^n\) if and only if, for the matrix \(A = [v_1, v_2, \ldots, v_k]\), the linear system \(A\mathbf{c} = \mathbf{v}\) is consistent for every \(\mathbf{v}\) in \(\mathbb{R}^n\).
4.4 Spanning Sets

Proof: Rewriting the system \( A\mathbf{v} = \mathbf{v} \) as the linear combination
\[ c_1 y_1 + c_2 y_2 + \cdots + c_l y_l = \mathbf{v}, \]
we see that the existence of a solution \((c_1, c_2, \ldots, c_l)\) to this vector equation for each \( \mathbf{v} \) in \( \mathbb{R}^n \) is equivalent to the statement that \( \{y_1, y_2, \ldots, y_l\} \) spans \( \mathbb{R}^n \).

Next, we consider a couple of examples involving vector spaces other than \( \mathbb{R}^n \).

Example 4.4.5

Verify that
\[
A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \quad A_4 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}
\]
span \( M_2(\mathbb{R}) \).

Solution: An arbitrary vector in \( M_2(\mathbb{R}) \) is of the form
\[
A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.
\]
If we write
\[
c_1 A_1 + c_2 A_2 + c_3 A_3 + c_4 A_4 = A,
\]
then equating the elements of the matrices on each side of the equation yields the system
\[
c_1 + c_2 + c_3 + c_4 = a,
\]
\[
c_2 + c_3 + c_4 = b,
\]
\[
c_3 + c_4 = c,
\]
\[
c_4 = d.
\]
Solving this by back substitution gives
\[
c_1 = a - b, \quad c_2 = b - c, \quad c_3 = c - d, \quad c_4 = d.
\]
Hence, we have
\[
A = (a - b)A_1 + (b - c)A_2 + (c - d)A_3 + dA_4.
\]
Consequently every vector in \( M_2(\mathbb{R}) \) can be written as a linear combination of \( A_1, A_2, A_3, \) and \( A_4, \) and therefore these matrices do indeed span \( M_2(\mathbb{R}) \).

Remark: The most natural spanning set for \( M_2(\mathbb{R}) \) is
\[
\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\},
\]
a fact that we leave to the reader as an exercise.

Example 4.4.6

Determine a spanning set for \( P_2 \), the vector space of all polynomials of degree 2 or less.

Solution: The general polynomial in \( P_2 \) is
\[
p(x) = a_0 + a_1 x + a_2 x^2.
\]
If we let
\[
p_0(x) = 1, \quad p_1(x) = x, \quad p_2(x) = x^2,
\]
then
\[
p(x) = a_0 p_0(x) + a_1 p_1(x) + a_2 p_2(x).
\]
Thus, every vector in \( P_2 \) is a linear combination of \( 1, x, \) and \( x^2 \), and so a spanning set for \( P_2 \) is \( \{1, x, x^2\} \). For practice, the reader might show that \( \{x^2, x + x^2, 1 + x + x^2\} \) is another spanning set for \( P_2 \), by making the appropriate modifications to the calculations in this example.
The Linear Span of a Set of Vectors

Now let \(v_1, v_2, \ldots, v_k\) be vectors in a vector space \(V\). Forming all possible linear combinations of \(v_1, v_2, \ldots, v_k\) generates a subset of \(V\) called the **linear span of** \(\{v_1, v_2, \ldots, v_k\}\), denoted span\(\{v_1, v_2, \ldots, v_k\}\). We have

\[
\text{span}(v_1, v_2, \ldots, v_k) = \{ v \in V : v = c_1 v_1 + c_2 v_2 + \cdots + c_k v_k, c_1, c_2, \ldots, c_k \in F \}.
\]

(4.4.5)

For example, suppose \(V = C^2(\mathbb{R})\), and let \(y_1(x) = \sin x\) and \(y_2(x) = \cos x\). Then

\[
\text{span}(y_1, y_2) = \{ y \in C^2(\mathbb{R}) : y(x) = c_1 \cos x + c_2 \sin x, c_1, c_2 \in \mathbb{R} \}.
\]

From Example 1.2.16, we recognize \(y_1\) and \(y_2\) as being nonproportional solutions to the differential equation \(y'' + y = 0\). Consequently, in this example, the linear span of the given functions coincides with the set of all solutions to the differential equation \(y'' + y = 0\) and therefore is a subspace of \(V\). Our next theorem generalizes this to show that any linear span of vectors in any vector space forms a subspace.

**Theorem 4.4.7**

Let \(v_1, v_2, \ldots, v_k\) be vectors in a vector space \(V\). Then span\(\{v_1, v_2, \ldots, v_k\}\) is a subspace of \(V\).

**Proof**

Let \(S = \text{span}\{v_1, v_2, \ldots, v_k\}\). Then \(0 \in S\) (corresponding to \(c_1 = c_2 = \cdots = c_k = 0\) in (4.4.5)), so \(S\) is nonempty. We now verify closure of \(S\) under addition and scalar multiplication. If \(u, v \in S\), then, from Equation (4.4.5),

\[
u = a_1 v_1 + a_2 v_2 + \cdots + a_k v_k \quad \text{and} \quad v = b_1 v_1 + b_2 v_2 + \cdots + b_k v_k,
\]

for some scalars \(a_i, b_i\). Thus,

\[
u + v = (a_1 b_1 v_1 + a_2 b_2 v_2 + \cdots + a_k b_k v_k)
\]

and

\[
u + v = (a_1 b_1 v_1 + a_2 b_2 v_2 + \cdots + a_k b_k v_k)
\]

where \(c_i = a_i + b_i\) for each \(i = 1, 2, \ldots, k\). Consequently, \(u + v\) has the proper form for membership in \(S\) according to (4.4.5), so \(S\) is closed under addition. Further, if \(r\) is any scalar, then

\[
r \nu = r(a_1 v_1 + a_2 v_2 + \cdots + a_k v_k)
\]

and

\[
r \nu = r(a_1 v_1 + a_2 v_2 + \cdots + a_k v_k)
\]

where \(d_i = r c_i\) for each \(i = 1, 2, \ldots, k\). Consequently, \(r \nu \in S\), and so \(S\) is also closed under scalar multiplication. Hence, \(S = \text{span}\{v_1, v_2, \ldots, v_k\}\) is a subspace of \(V\). \(\blacksquare\)

**Remarks**

1. We will also refer to \(\text{span}\{v_1, v_2, \ldots, v_k\}\) as the **subspace of** \(V\) spanned by \(v_1, v_2, \ldots, v_k\).

2. As a special case, we will declare that \(\text{span}(\emptyset) = \{0\}\).
4.4 Spanning Sets

**Example 4.4.8**

If \( V = \mathbb{R}^2 \) and \( v_1 = (-1, 1) \), determine \( \text{span}(v_1) \).

**Solution:**

\[
\text{span}(v_1) = \{ v \in \mathbb{R}^2 : v = c_1 v_1, \ c_1 \in \mathbb{R} \} = \{ v \in \mathbb{R}^2 : v = (c_1, c_1), \ c_1 \in \mathbb{R} \}.
\]

Geometrically, this is the line through the origin with parametric equations \( x = -c_1, \ y = c_1 \), so that the Cartesian equation of the line is \( y = -x \). (See Figure 4.4.4.)

![Figure 4.4.4: The subspace of \( \mathbb{R}^2 \) spanned by \( v_1 = (-1, 1) \).](image)

**Example 4.4.9**

If \( V = \mathbb{R}^3, v_1 = (1, 0, 1), \) and \( v_2 = (0, 1, 1) \), determine the subspace of \( \mathbb{R}^3 \) spanned by \( v_1 \) and \( v_2 \). Does \( w = (1, 1, -1) \) lie in this subspace?

**Solution:**

\[
\text{span}(v_1, v_2) = \{ v \in \mathbb{R}^3 : v = c_1 v_1 + c_2 v_2, \ c_1, c_2 \in \mathbb{R} \} = \{ v \in \mathbb{R}^3 : v = c_1(1, 0, 1) + c_2(0, 1, 1), \ c_1, c_2 \in \mathbb{R} \} = \{ v \in \mathbb{R}^3 : v = (c_1, c_2, c_1 + c_2), \ c_1, c_2 \in \mathbb{R} \}.
\]

Since the vector \( w = (1, 1, -1) \) is not of the form \((c_1, c_2, c_1 + c_2)\), it does not lie in \( \text{span}(v_1, v_2) \). Geometrically, \( \text{span}(v_1, v_2) \) is the plane through the origin determined by the two given vectors \( v_1 \) and \( v_2 \). It has parametric equations \( x = c_1, \ y = c_2, \ z = c_1 + c_2 \), which implies that its Cartesian equation is \( z = x + y \). Thus, the fact that \( w \) is not in \( \text{span}(v_1, v_2) \) means that \( w \) does not lie in this plane. The subspace is depicted in Figure 4.4.5.

![Figure 4.4.5: The subspace of \( \mathbb{R}^3 \) spanned by \( v_1 = (1, 0, 1) \) and \( v_2 = (0, 1, 1) \) is the plane with Cartesian equation \( z = x + y \).](image)
Example 4.4.10  
Let 
\[ A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \]
in \( M_2(\mathbb{R}) \). Determine \( \text{span} \{ A_1, A_2, A_3 \} \).

**Solution:**  By definition we have 
\[ \text{span} \{ A_1, A_2, A_3 \} = \{ A \in M_2(\mathbb{R}) : A = a_1 A_1 + a_2 A_2 + a_3 A_3, \; a_1, a_2, a_3 \in \mathbb{R} \} \]
\[ = \{ A \in M_2(\mathbb{R}) : A = a_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + a_2 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + a_3 \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \} \]
\[ = \{ A \in M_2(\mathbb{R}) : A = \begin{bmatrix} c_1 & c_2 \\ c_2 & c_3 \end{bmatrix}, \; c_1, c_2, c_3 \in \mathbb{R} \} \]
This is the set of all real \( 2 \times 2 \) symmetric matrices. \( \square \)

**Example 4.4.11**  
Determine the subspace of \( P_2 \) spanned by 
\[ p_1(x) = 1 + 3x, \quad p_2(x) = x + x^2, \]
and decide whether \( \{ p_1, p_2 \} \) is a spanning set for \( P_2 \).

**Solution:**  We have 
\[ \text{span} \{ p_1, p_2 \} = \{ p \in P_2 : p(x) = c_1 p_1(x) + c_2 p_2(x), \; c_1, c_2 \in \mathbb{R} \} \]
\[ = \{ p \in P_2 : p(x) = c_1 (1 + 3x) + c_2 (x + x^2), \; c_1, c_2 \in \mathbb{R} \} \]
\[ = \{ p \in P_2 : p(x) = c_1 + (3c_1 + c_2)x + c_2x^2, \; c_1, c_2 \in \mathbb{R} \} \]
Next, we will show that \( \{ p_1, p_2 \} \) is not a spanning set for \( P_2 \). To establish this, we need give only one example of a polynomial in \( P_2 \) that is not in \( \text{span} \{ p_1, p_2 \} \). There are many such choices here, but suppose we consider \( p(x) = 1 + x \). If this polynomial were in \( \text{span} \{ p_1, p_2 \} \), then we would have to be able to find values of \( c_1 \) and \( c_2 \) such that 
\[ 1 + x = c_1 + (3c_1 + c_2)x + c_2x^2. \]  \( (4.4.6) \)
Since there is no \( x^2 \) term on the left-hand side of this expression, we must set \( c_2 = 0 \). But then \( (4.4.6) \) would reduce to 
\[ 1 + x = c_1 (1 + 3x) \]
Equating the constant terms on each side of this forces \( c_1 = 1 \), but then the coefficients of \( x \) do not match. Hence, such an equality is impossible. Consequently, there are no values of \( c_1 \) and \( c_2 \) such that the Equation \( (4.4.6) \) holds, and therefore, \( \text{span} \{ p_1, p_2 \} \neq P_2 \). \( \square \)

**Remark**  In the previous example, the reader may well wonder why we knew from the beginning to select \( p(x) = 1 + x \) as a vector that would be outside of \( \text{span} \{ p_1, p_2 \} \). In truth, we only need to find a polynomial that does not have the form \( p(x) = c_1 + (3c_1 + c_2)x + c_2x^2 \) and in fact, “most” of the polynomials in \( P_2 \) would have achieved the desired result here.
Exercises for 4.4

**Key Terms**

Linear combination, Linear span, Spanning set.

**Skills**

- Be able to determine whether a given set of vectors $S$ spans a vector space $V$, and be able to prove your answer mathematically.
- Be able to determine the linear span of a set of vectors. For vectors in $\mathbb{R}^n$, be able to give a geometric description of the linear span.
- If $S$ is a spanning set for a vector space $V$, be able to write any vector in $V$ as a linear combination of the elements of $S$.
- Be able to construct a spanning set for a vector space $V$. As a special case, be able to determine a spanning set for the null space of an $m \times n$ matrix.
- Be able to determine whether a particular vector $v$ in a vector space $V$ lies in the linear span of a set $S$ of vectors in $V$.
- Every vector space $V$ has a finite spanning set.

**True-False Review**

For Questions 1–12, decide if the given statement is true or false, and give a brief justification for your answer. If true, you can quote a relevant definition or theorem from the text. If false, provide an example, illustration, or brief explanation of why the statement is false.

1. The linear span of a set of vectors in a vector space $V$ forms a subspace of $V$.
2. If some vector $v$ in a vector space $V$ is a linear combination of vectors in a set $S$, then $S$ spans $V$.
3. If $S$ is a spanning set for a vector space $V$ and $W$ is a subspace of $V$, then $S$ spans $W$.
4. If $S$ is a spanning set for a vector space $V$, then every vector $v$ in $V$ must be uniquely expressible as a linear combination of the vectors in $S$.
5. A set $S$ of vectors in a vector space $V$ spans $V$ if and only if the linear span of $S$ is $V$.
6. The linear span of two vectors in $\mathbb{R}^3$ is a plane through the origin.
7. Every vector space $V$ has a finite spanning set.
8. If $S$ is a spanning set for a vector space $V$, then any proper subset $S'$ of $S$ is not a spanning set for $V$.
9. The vector space of $3 \times 3$ upper triangular matrices is spanned by the matrices $E_{ij}$ where $1 \leq i \leq j \leq 3$.
10. A spanning set for the vector space $P_2$ must contain a polynomial of each degree 0, 1, and 2.
11. If $m < n$, then any spanning set for $\mathbb{R}^m$ must contain more vectors than any spanning set for $\mathbb{R}^n$.
12. The vector space $P$ of all polynomials with real coefficients cannot be spanned by a finite set $S$.

**Problems**

For Problems 1–3, determine whether the given set of vectors spans $\mathbb{R}^2$.

1. $\{(1, -1), (2, -2), (2, 3)\}$.
2. $\{(2, 5), (0, 0)\}$.
3. $\{(-2, -1, 1), (3, 2, 1)\}$.

Recall that three vectors $v_1, v_2, v_3$ in $\mathbb{R}^3$ are coplanar if and only if $\det([v_1, v_2, v_3]) = 0$.

For Problems 4–6, use this result to determine whether the given set of vectors spans $\mathbb{R}^3$.

4. $\{(1, -1, 1), (2, 5, 3), (4, -2, 1)\}$.
5. $\{(1, -2, 1), (2, 3, 1), (0, 0, 0), (4, -1, 2)\}$.
6. $\{(2, -1, 4), (3, -3, 5), (1, 1, 3)\}$.

7. Show that the set of vectors
   $$\{(-2, 3, 1, 2), (3, 4, 5, 6)\}$$
   does not span $\mathbb{R}^3$, but that it does span the subspace of $\mathbb{R}^3$ consisting of all vectors lying in the plane with equation $x - 2y + z = 0$.

8. Show that $v_1 = (2, -1, 2)$, $v_2 = (3, 2)$ span $\mathbb{R}^2$, and express the vector $v = (5, -7)$ as a linear combination of $v_1, v_2$.

9. Show that $v_1 = (-1, 3, 2), v_2 = (1, -2, 1), v_3 = (2, 1, 1)$ span $\mathbb{R}^3$, and express $v = (x, y, z)$ as a linear combination of $v_1, v_2, v_3$. 
10. Show that \( v_1 = (1, 1), v_2 = (-1, 2), v_3 = (1, 4) \) span \( \mathbb{R}^2 \). Do \( v_1, v_2 \) alone span \( \mathbb{R}^2 \) also?

11. Let \( S \) be the subspace of \( \mathbb{R}^3 \) consisting of all vectors of the form \( v = (c_1, c_2, c_2 - 2c_1) \). Show that \( S \) is spanned by \( v_1 = (1, 0, -2), v_2 = (0, 1, 1) \).

12. Let \( S \) be the subspace of \( \mathbb{R}^4 \) consisting of all vectors of the form \( v = (c_1, c_2, c_2 - c_1 - c_1, 2c_2) \). Determine a set of vectors that spans \( S \).

13. Let \( S \) be the subspace of \( \mathbb{R}^3 \) consisting of all solutions to the linear system
\[
\begin{align*}
x - 2y - z &= 0.
\end{align*}
\]
Determine a set of vectors that spans \( S \).

For Problems 14–15, determine a spanning set for the null space of the given matrix \( A \).

14. \( A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 5 & 6 & 7 \end{bmatrix} \).

15. \( A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 2 \\ 2 & 4 & 6 \end{bmatrix} \).

16. Let \( S \) be the subspace of \( M_2(\mathbb{R}) \) consisting of all symmetric \( 2 \times 2 \) matrices with real elements. Show that \( S \) is spanned by the matrices
\[
\begin{align*}
A_1 &= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, & A_2 &= \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, & A_3 &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.
\end{align*}
\]

17. Let \( S \) be the subspace of \( M_2(\mathbb{R}) \) consisting of all skew-symmetric \( 2 \times 2 \) matrices with real elements. Determine a matrix that spans \( S \).

18. Let \( S \) be the subset of \( M_2(\mathbb{R}) \) consisting of all upper triangular \( 2 \times 2 \) matrices.

(a) Verify that \( S \) is a subspace of \( M_2(\mathbb{R}) \).

(b) Determine a set of \( 2 \times 2 \) matrices that spans \( S \).

For Problems 19–20, determine \( \text{span}\{v_1, v_2\} \) for the given vectors in \( \mathbb{R}^3 \), and describe it geometrically.

19. \( v_1 = (1, -1, 2), v_2 = (2, -1, 3) \).

20. \( v_1 = (1, 2, -1), v_2 = (-2, -4, 2) \).

21. Let \( S \) be the subspace of \( \mathbb{R}^3 \) spanned by the vectors
\[
\begin{align*}
v_1 &= (1, 1, -1), v_2 = (2, 1, 3), v_3 = (-2, -2, 2).
\end{align*}
\]
Show that \( S \) also is spanned by \( v_1 \) and \( v_2 \) only.

For Problems 22–24, determine whether the given vector \( v \) lies in \( \text{span}\{v_1, v_2\} \).

22. \( v = (3, 3, 4), v_1 = (1, -1, 2), v_2 = (2, 1, 3) \) in \( \mathbb{R}^3 \).

23. \( v = (5, 3, -6), v_1 = (-1, 1, 2), v_2 = (3, 1, -4) \) in \( \mathbb{R}^3 \).

24. \( v = (1, 1, -2), v_1 = (3, 1, 2), v_2 = (-2, -1, 1) \) in \( \mathbb{R}^3 \).

25. If \( p_1(x) = x - 4 \) and \( p_2(x) = x^2 - x + 3 \), determine whether \( p(x) = 2x^2 - x + 2 \) lies in \( \text{span}\{p_1, p_2\} \).

26. Consider the vectors
\[
\begin{align*}
A_1 &= \begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix}, & A_2 &= \begin{bmatrix} 0 & 1 \\ -2 & 1 \end{bmatrix}, & A_3 &= \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}
\end{align*}
\]
in \( M_2(\mathbb{R}) \). Determine \( \text{span}\{A_1, A_2, A_3\} \).

27. Consider the vectors
\[
\begin{align*}
A_1 &= \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}, & A_2 &= \begin{bmatrix} -2 & 1 \\ 1 & -1 \end{bmatrix}
\end{align*}
\]
in \( M_2(\mathbb{R}) \). Find \( \text{span}\{A_1, A_2\} \), and determine whether or not
\[
B = \begin{bmatrix} 3 & 1 \\ -2 & 4 \end{bmatrix}
\]
lies in this subspace.

28. Let \( V = C^\infty(\mathbb{I}) \) and let \( S \) be the subspace of \( V \) spanned by the functions
\[
f(x) = \cosh x, \quad g(x) = \sinh x.
\]
(a) Give an expression for a general vector in \( S \).

(b) Show that \( S \) is also spanned by the functions
\[
h(x) = e^x, \quad j(x) = e^{-x}.
\]

For Problems 29–32, give a geometric description of the subspace of \( \mathbb{R}^3 \) spanned by the given set of vectors.

29. \( \{0\} \).

30. \( \{v_1\} \), where \( v_1 \) is any nonzero vector in \( \mathbb{R}^3 \).

31. \( \{v_1, v_2\} \), where \( v_1, v_2 \) are nonzero and noncollinear vectors in \( \mathbb{R}^3 \).
Linear Dependence and Linear Independence

32. \{v_1, v_2\}, where \(v_1, v_2\) are collinear vectors in \(\mathbb{R}^3\).

33. Prove that if \(S\) and \(S'\) are subsets of a vector space \(V\) such that \(S\) is a subset of \(S'\), then \(\text{span}(S)\) is a subset of \(\text{span}(S')\).

34. Prove that \(\text{span}\{v_1, v_2, v_3\} = \text{span}\{v_1, v_2\}\) if and only if \(v_3\) can be written as a linear combination of \(v_1\) and \(v_2\).

4.5 Linear Dependence and Linear Independence

As indicated in the previous section, in analyzing a vector space we will be interested in determining a spanning set. The reader has perhaps already noticed that a vector space \(V\) can have many such spanning sets.

Example 4.5.1

Observe that \((1, 0), (0, 1), (1, 1), \) and \((1, 0), (0, 1), (1, 2)\) are all spanning sets for \(\mathbb{R}^2\).

As another illustration, two different spanning sets for \(V = M_2(\mathbb{R})\) were given in Example 4.4.5 and the remark that followed. Given the abundance of spanning sets available for a given vector space \(V\), we are faced with a natural question: Is there a "best class of" spanning sets to use? The answer, to a large degree, is "yes". For instance, in Example 4.5.1, the spanning set \((1, 0), (0, 1), (1, 2)\) contains an "extra" vector, \((1, 2)\), which seems to be unnecessary for spanning \(\mathbb{R}^2\), since \((1, 0), (0, 1)\) is already a spanning set. In some sense, \((1, 0), (0, 1)\) is a more efficient spanning set. It is what we call a minimal spanning set, since it contains the minimum number of vectors needed to span the vector space.

But how will we know if we have found a minimal spanning set (assuming one exists)? Returning to the example above, we have seen that

\[\text{span}((1, 0), (0, 1)) = \text{span}((1, 0), (0, 1), (1, 2)) = \mathbb{R}^2.\]

Observe that the vector \((1, 2)\) is already a linear combination of \((1, 0)\) and \((0, 1)\), and therefore it does not add any new vectors to the linear span of \((1, 0), (0, 1)\).

As a second example, consider the vectors \(v_1 = (1, 1, 1), v_2 = (3, -2, 1),\) and \(v_3 = 2v_1 + v_2 = (7, 2, 5).\) It is easily verified that \(\det([v_1, v_2, v_3]) = 0.\) Consequently, the three vectors lie in a plane (see Figure 4.5.1) and therefore, since they are not collinear, the linear span of these three vectors is the whole of this plane. Furthermore, the same plane is generated if we consider the linear span of \(v_1\) and \(v_2\) alone. As in the previous example, the reason that \(v_3\) does not add any new vectors to the linear span of \([v_1, v_2]\) is that it is already a linear combination of \(v_1\) and \(v_2\). It is not possible, however, to generate all vectors in the plane by taking linear combinations of just one vector, as we could generate only a line lying in the plane in that case. Consequently, \([v_1, v_2]\) is a minimal spanning set for the subspace of \(\mathbb{R}^3\) consisting of all points lying on the plane.

As a final example, recall from Example 1.2.16 that the solution space to the differential equation

\[y'' + y = 0\]

\(3\)Since a single (nonzero) vector in \(\mathbb{R}^3\) spans only the line through the origin along which it points, it cannot span all of \(\mathbb{R}^3\), hence, the minimum number of vectors required to span \(\mathbb{R}^3\) is 2.