Areas of Surfaces of Revolution

When you jump rope, the rope sweeps out a surface in the space around you, a surface called a surface of revolution. As you can imagine, the area of this surface depends on the rope’s length and on how far away each segment of the rope swings. This section explores the relation between the area of a surface of revolution and the length and reach of the curve that generates it. The areas of more complicated surfaces will be treated in Chapter 14.

The Basic Formula

Suppose we want to find the area of the surface swept out by revolving the graph of a nonnegative function \( y = f(x), \ a \leq x \leq b \), about the \( x \)-axis. We partition \([a, b] \) in the usual way and use the points in the partition to partition the graph into short arcs. Figure 5.41 shows a typical arc \( PQ \) and the band it sweeps out as part of the graph of \( f \).

As the arc \( PQ \) revolves about the \( x \)-axis, the line segment joining \( P \) and \( Q \) sweeps out part of a cone whose axis lies along the \( x \)-axis (magnified view in Fig. 5.42). A piece of a cone like this is called a frustum of the cone, frustum being Latin for “piece.” The surface area of the frustum approximates the surface area of the band swept out by the arc \( PQ \).

The surface area of the frustum of a cone (see Fig. 5.43) is \( 2\pi \) times the average of the base radii times the slant height:

\[
\text{Frustum surface area} = 2\pi \cdot \frac{r_1 + r_2}{2} \cdot L = \pi (r_1 + r_2)L.
\]

For the frustum swept out by the segment \( PQ \) (Fig. 5.44), this works out to be

\[
\text{Frustum surface area} = \pi (f(x_{k-1}) + f(x_k))\sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}.
\]

The area of the original surface, being the sum of the areas of the bands swept out by arcs like arc \( PQ \), is approximated by the frustum area sum

\[
\sum_{k=1}^{n} \pi (f(x_{k-1}) + f(x_k))\sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}.
\] (1)

We expect the approximation to improve as the partition of \([a, b] \) becomes finer, and we would like to show that the sums in (1) approach a calculable limit as the norm of the partition goes to zero.
To show this, we try to rewrite the sum in (1) as the Riemann sum of some function over the interval from \(a\) to \(b\). As in the calculation of arc length, we begin by appealing to the Mean Value Theorem for derivatives.

If \(f\) is smooth, then by the Mean Value Theorem, there is a point \((c_k, f(c_k))\) on the curve between \(P\) and \(Q\) where the tangent is parallel to the segment \(PQ\) (Fig. 5.45). At this point,

\[
f'(c_k) = \frac{\Delta y_k}{\Delta x_k},
\]

\[
\Delta y_k = f'(c_k) \Delta x_k.
\]

With this substitution for \(\Delta y_k\), the sums in (1) take the form

\[
\sum_{k=1}^{n} \pi \left[ f(x_{k-1}) + f(x_k) \right] \sqrt{(\Delta x_k)^2 + (f'(c_k) \Delta x_k)^2}
\]

\[
= \sum_{k=1}^{n} \pi \left[ f(x_{k-1}) + f(x_k) \right] \sqrt{1 + (f'(c_k))^2} \Delta x_k. \tag{2}
\]

At this point there is both good news and bad news.

The bad news is that the sums in (2) are not the Riemann sums of any function because the points \(x_{k-1}, x_k\), and \(c_k\) are not the same and there is no way to make them the same. The good news is that this does not matter. A theorem called Bliss’s theorem, from advanced calculus, assures us that as the norm of the partition of \([a, b]\) goes to zero, the sums in Eq. (2) converge to

\[
\int_a^b 2\pi f(x) \sqrt{1 + (f'(x))^2} \, dx
\]

just the way we want them to. We therefore define this integral to be the area of the surface swept out by the graph of \(f\) from \(a\) to \(b\).

**Definition**

**The Surface Area Formula for the Revolution About the \(x\)-axis**

If the function \(f(x) \geq 0\) is smooth on \([a, b]\), the area of the surface generated by revolving the curve \(y = f(x)\) about the \(x\)-axis is

\[
S = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx = \int_a^b 2\pi f(x) \sqrt{1 + (f'(x))^2} \, dx. \tag{3}
\]
The square root in Eq. (3) is the same one that appears in the formula for the length of the generating curve.

**EXAMPLE 1** Find the area of the surface generated by revolving the curve $y = 2\sqrt{x}$, $1 \leq x \leq 2$, about the $x$-axis (Fig. 5.46).

**Solution** We evaluate the formula

$$S = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx \quad \text{Eq. (3)}$$

with

$$a = 1, \quad b = 2, \quad y = 2\sqrt{x}, \quad \frac{dy}{dx} = \frac{1}{\sqrt{x}},$$

$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + \left(\frac{1}{\sqrt{x}}\right)^2} = \sqrt{1 + \frac{1}{x}} = \sqrt{\frac{x + 1}{x}} = \frac{\sqrt{x + 1}}{\sqrt{x}}.$$

With these substitutions,

$$S = \int_1^2 2\pi \cdot 2\sqrt{x} \frac{\sqrt{x + 1}}{\sqrt{x}} \, dx = 4\pi \int_1^2 \frac{x + 1}{\sqrt{x}} \, dx = 4\pi \left[ \frac{2}{3} (x + 1)^{3/2} \right]_1^2 = \frac{8\pi}{3} (3\sqrt{3} - 2\sqrt{2}).$$

**Revolution About the $y$-axis**

For revolution about the $y$-axis, we interchange $x$ and $y$ in Eq. (3).

**Surface Area Formula for Revolution About the $y$-axis**

If $x = g(y) \geq 0$ is smooth on $[c, d]$, the area of the surface generated by revolving the curve $x = g(y)$ about the $y$-axis is

$$S = \int_c^d 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2} \, dy = \int_c^d 2\pi g(y) \sqrt{1 + (g'(y))^2} \, dy. \quad (4)$$

**EXAMPLE 2** The line segment $x = 1 - y$, $0 \leq y \leq 1$, is revolved about the $y$-axis to generate the cone in Fig. 5.47. Find its lateral surface area.

**Solution** Here we have a calculation we can check with a formula from geometry:

$$\text{Lateral surface area} = \frac{\text{base circumference}}{2} \times \text{slant height} = \pi \sqrt{2}.$$
To see how Eq. (4) gives the same result, we take
\[ c = 0, \quad d = 1, \quad x = 1 - y, \quad \frac{dx}{dy} = -1, \]
\[ \sqrt{1 + \left( \frac{dx}{dy} \right)^2} = \sqrt{1 + (-1)^2} = \sqrt{2} \]
and calculate
\[ S = \int_c^d 2\pi x \sqrt{1 + \left( \frac{dx}{dy} \right)^2} \, dy = \int_0^1 2\pi (1 - y)\sqrt{2} \, dy \]
\[ = 2\pi \sqrt{2} \left[ y - \frac{y^2}{2} \right]_0^1 = 2\pi \sqrt{2} \left( 1 - \frac{1}{2} \right) \]
\[ = \pi \sqrt{2}. \]

The results agree, as they should.

**The Short Differential Form**

The equations
\[ S = \int_a^b 2\pi y \sqrt{1 + \left( \frac{dy}{dx} \right)^2} \, dx \quad \text{and} \quad S = \int_c^d 2\pi x \sqrt{1 + \left( \frac{dx}{dy} \right)^2} \, dy \]
are often written in terms of the arc length differential \( ds = \sqrt{dx^2 + dy^2} \) as
\[ S = \int_a^b 2\pi y \, ds \quad \text{and} \quad S = \int_c^d 2\pi x \, ds. \]

In the first of these, \( y \) is the distance from the \( x \)-axis to an element of arc length \( ds \). In the second, \( x \) is the distance from the \( y \)-axis to an element of arc length \( ds \). Both integrals have the form
\[ S = \int 2\pi (\text{radius})(\text{band width}) = \int 2\pi \rho \, ds, \]
where \( \rho \) is the radius from the axis of revolution to an element of arc length \( ds \) (Fig. 5.48).
Short Differential Form

\[ S = \int 2\pi \rho \, ds \]

In any particular problem, you would then express the radius function \( \rho \) and the arc length differential \( ds \) in terms of a common variable and supply limits of integration for that variable.

**EXAMPLE 3**  Find the area of the surface generated by revolving the curve \( y = x^3, \ 0 \leq x \leq 1/2 \), about the \( x \)-axis (Fig. 5.49).

**Solution**  We start with the short differential form:

\[
S = \int 2\pi \rho \, ds \\
= \int 2\pi y \, ds \\
= \int 2\pi y \sqrt{dx^2 + dy^2}.
\]

For revolution about the \( x \)-axis, the radius function is \( \rho = y \).

\( ds = \sqrt{dx^2 + dy^2} \)

We then decide whether to express \( dy \) in terms of \( dx \) or \( dx \) in terms of \( dy \). The original form of the equation, \( y = x^3 \), makes it easier to express \( dy \) in terms of \( dx \), so we continue the calculation with

\[
y = x^3, \quad dy = 3x^2 \, dx, \quad \text{and} \quad \sqrt{dx^2 + dy^2} = \sqrt{dx^2 + (3x^2 \, dx)^2} = \sqrt{1 + 9x^4 \, dx}.
\]

With these substitutions, \( x \) becomes the variable of integration and

\[
S = \int_{x=0}^{x=1/2} 2\pi y \sqrt{dx^2 + dy^2} \\
= \int_{0}^{1/2} 2\pi x^3 \sqrt{1 + 9x^4} \, dx \\
= 2\pi \left( \frac{1}{36} \right) \left( \frac{2}{3} \right) (1 + 9x^4)^{3/2} \bigg|_{0}^{1/2} \quad \text{Substitute } u = 1 + 9x^4, \, du/36 = x^3 \, dx, \text{ integrate, and substitute back.}
\]

\[
= \pi \frac{\left[ (1 + 9 \frac{1}{16})^{3/2} - 1 \right]}{27} = \pi \frac{\left[ (25/16)^{3/2} - 1 \right]}{27} = \pi \left( \frac{125}{64} - 1 \right)
\]

\[
= \frac{61\pi}{1728}.
\]

As with arc length calculations, even the simplest curves can provide a workout.
15. \( y = \sqrt{2x - x^2}, \quad 0.5 \leq x \leq 1.5; \) x-axis
16. \( y = \sqrt{x + 1}, \quad 1 \leq x \leq 5; \) x-axis
17. \( x = y^3/3, \quad 0 \leq y \leq 1; \) y-axis
18. \( x = (1/3)y^{3/2} - y^{1/2}, \quad 1 \leq y \leq 3; \) y-axis
19. \( x = 2\sqrt{4 - y}, \quad 0 \leq y \leq 15/4; \) y-axis

20. \( x = \sqrt{2y - 1}, \quad 5/8 \leq y \leq 1; \) y-axis

21. \( x = (y^4/4) + 1/(8y^2), \quad 1 \leq y \leq 2; \) x-axis (Hint: Express \( ds = \sqrt{dx^2 + dy^2} \) in terms of \( dy \), and evaluate the integral \( S = \int 2\pi y \, ds \) with appropriate limits.)
22. \( y = (1/3)(x^2 + 2)^{3/2}, \quad 0 \leq x \leq \sqrt{2}; \) y-axis (Hint: Express \( ds = \sqrt{dx^2 + dy^2} \) in terms of \( dx \), and evaluate the integral \( S = \int 2\pi x \, ds \) with appropriate limits.)

23. Testing the new definition. Show that the surface area of a sphere of radius \( a \) is still \( 4\pi a^2 \) by using Eq. (3) to find the area of the surface generated by revolving the curve \( y = \sqrt{a^2 - x^2}, \quad -a \leq x \leq a \), about the x-axis.

24. Testing the new definition. The lateral (side) surface area of a cone of height \( h \) and base radius \( r \) should be \( \pi r \sqrt{r^2 + h^2} \), the semiperimeter of the base times the slant height. Show that this is still the case by finding the area of the surface generated by revolving the line segment \( y = (r/h)x, \quad 0 \leq x \leq h \), about the x-axis.

25. a) Write an integral for the area of the surface generated by revolving the curve \( y = \cos x, \quad -\pi/2 \leq x \leq \pi/2 \), about the x-axis. In Section 7.4 we will see how to evaluate such integrals.

b) CALCULATOR Find the surface area numerically.
26. The surface of an astroid. Find the area of the surface generated by revolving about the x-axis the portion of the astroid $x^{2/3} + y^{2/3} = 1$ shown here. (Hint: Revolve the first-quadrant portion $y = (1 - x^{2/3})^{3/2}$, $0 \leq x \leq 1$, about the x-axis and double your result.)

27. Enameling woks. Your company decided to put out a deluxe version of the successful wok you designed in Section 5.3, Exercise 41. The plan is to coat it inside with white enamel and outside with blue enamel. Each enamel will be sprayed on 0.5 mm thick before baking. (See diagram here.) Your manufacturing department wants to know how much enamel to have on hand for a production run of 5000 woks. What do you tell them? (Neglect waste and unused material and give your answer in liters. Remember that 1 cm³ = 1 mL, so 1 L = 1000 cm³.)

28. Slicing bread. Did you know that if you cut a spherical loaf of bread into slices of equal width, each slice will have the same amount of crust? To see why, suppose the semicircle $y = \sqrt{r^2 - x^2}$ shown here is revolved about the x-axis to generate a sphere. Let AB be an arc of the semicircle that lies above an interval of length h on the x-axis. Show that the area swept out by AB does not depend on the location of the interval. (It does depend on the length of the interval.)