1. It is given that
\[ A = \begin{bmatrix} 1 & 0 & -2 & 1 & 3 \\ -1 & 1 & 5 & -1 & -3 \\ 0 & 2 & 6 & 0 & 1 \\ 1 & 1 & 1 & 1 & 4 \end{bmatrix}, \quad \text{rref}(A) = \begin{bmatrix} 1 & 0 & -2 & 1 & 0 \\ 0 & 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \]
and
\[ \text{rref}(A^T) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \]
(a) Find the rank of \( A \).
(b) Find the nullity of \( A \).
(c) Find a basis for the column space of \( A \). We require that you choose the vectors for the basis from the column vectors of \( A \).
(d) Find another basis for the column space of \( A \).
(e) Find a basis for the row space of \( A \). We require that you choose the vectors for the basis from the row vectors of \( A \).
(f) Find another basis for the row space of \( A \).
(g) Find a basis for the null space of \( A \).
(h) Find a basis for the orthogonal complement of the row space of \( A \).
(i) Write the third column of \( A \) as a linear combination of the other columns.

2. It is given that
\[ A = \begin{bmatrix} 1 & 2 & 1 & -1 \\ 2 & 4 & 1 & -4 \\ -1 & -2 & 0 & 3 \\ 0 & 0 & 1 & 2 \\ 3 & 6 & 2 & -5 \end{bmatrix}, \quad \text{rref}(A^T) = \begin{bmatrix} 1 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \]
and
\[ \text{rref}(A) = \begin{bmatrix} 1 & 2 & 0 & -3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \]
(a) Find the rank of \( A \).
(b) Find the nullity of \( A \).
(c) Find a basis for the column space of \( A \). We require that you choose the vectors for the basis from the column vectors of \( A \).
(d) Find another basis for the column space of \( A \).
(e) Find a basis for the row space of \( A \). We require that you choose the vectors for the basis from the row vectors of \( A \).
(f) Find another basis for the row space of $A$.

(g) Find a basis for the null space of $A$.

(h) Find a basis for the orthogonal complement of the row space of $A$.

(i) Is the vector $[2 \ 3 \ 0 \ 1]$ in the row space of $A$?

3. Find an equation relating $a$, $b$ and $c$ so that the linear system
\[
\begin{align*}
2x + 2y + 3z &= a \\
3x - y + 5z &= b \\
x - 3y + 2z &= c
\end{align*}
\]
is consistent for any values $a$, $b$ and $c$ which satisfy that equation.

4. Determine the values of $a$ so that the linear system
\[
\begin{align*}
x + y + z &= 2 \\
2x + 3y + z &= 5 \\
2x + 3y + (a^2 - 1)z &= a + 1
\end{align*}
\]
has (a) no solution, (b) a unique solution, and (c) infinitely many solutions.

5. Let $L : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation such that
\[
L \left( \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix}, \quad L \left( \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}, \quad L \left( \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} -1 \\ 2 \\ 4 \end{bmatrix}.
\]
Find the standard matrix for $L$.

6. Let $L : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation such that
\[
L \left( \begin{bmatrix} -2 \\ 1 \\ -2 \end{bmatrix} \right) = \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix}, \quad L \left( \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix} \right) = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}, \quad L \left( \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} -1 \\ 2 \\ 4 \end{bmatrix}.
\]
Find $L \left( \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \right)$.

7. Find the standard matrix of the linear transformation $L$ defined by
\[
L \left( \begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = \begin{bmatrix} 2x - z \\ x + 2y + z \\ 3x - y \end{bmatrix}.
\]
8. If $A$ is a $5 \times 5$ matrix and $\det A = 3$ find $A^2(\text{adj}A)^2$.

9. If $\text{adj} A = \begin{bmatrix} 28 & -21 & 11 \\ 24 & 12 & -12 \\ -8 & 6 & 14 \end{bmatrix}$ and $\det A > 0$ find $A^{-1}$.

10. Find a basis for $S = \{t^2 + 1, t^2 + 2t, 3t^2 + t - 1\}$. Does $6t^2 - 1$ belong to $\text{span} S$?

11. For what values of $d$ are the vectors $[1, 3, d], [1, 1, 0]$ and $[0, 1, 1]$ linearly independent?

12. Compute the inverse of the matrix $A = \begin{bmatrix} 2 & 0 & 1 \\ -2 & 1 & 3 \\ 2 & -1 & 1 \end{bmatrix}$.

13. Compute the inverse of the matrix $A = \begin{bmatrix} 2 & -2 & 1 \\ -3 & 2 & 0 \\ 4 & 1 & 1 \end{bmatrix}$.

14. Find ALL diagonal matrices similar to $A = \begin{bmatrix} 1 & 1 \\ -5 & 7 \end{bmatrix}$.

15. Consider the set of all pairs $(x, y)$ of real numbers, with the following operations:
   a) $(x, y) \oplus (w, z) = (x + w, y + z)$, for all pairs $(x, y), (w, z)$.
   b) $c \odot (x, y) = (cx, y)$, for all pairs $(x, y)$ and every number $c$.
   Is this a vector space? If not, state which conditions fail and show why they fail.

16. Suppose $A$ is a $3 \times 3$ matrix with eigenvalues $1, 3, 5$. What are the eigenvalues of $A^2$?

17. Use the Gram-Schmidt process to construct an orthonormal basis for the subspace $W$ of $\mathbb{R}^4$ spanned by
   \[
   \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -3 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 0 \\ -3 \end{bmatrix} \right\}.
   \]

18. Let $W$ be the subspace of $\mathbb{R}^3$ spanned by the vector $w = [1 \ 1 \ -1]$.
   (a) Find a basis for the orthogonal complement $W^\perp$ of $W$. 

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(b) Find an orthonormal basis for $W^\perp$.

19. Let $W$ be the subspace of $\mathbb{R}^4$ spanned by the set of vectors \[
\begin{bmatrix}
1 \\ 2 \\ 0 \\ -2
\end{bmatrix}, \quad \begin{bmatrix}
0 \\ 1 \\ 2 \\ 1
\end{bmatrix}, \quad \begin{bmatrix}
4 \\ -1 \\ 0 \\ 1
\end{bmatrix},
\]
and $v = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 4 \end{bmatrix}$. (a) Find the projection of $v$ onto $W$. (b) Find the distance from $v$ to $W$.

20. Determine if each of the following statements is true for every $5 \times 7$ matrix $A$ with rank($A$) = 3.

(a) For every $b$, the system $Ax = b$ is uniquely solvable. True False
(b) For some $b$, the system $Ax = b$ is uniquely solvable, and for some $b$, it is not solvable. True False
(c) For every $b$, the system $Ax = b$ has infinitely many solutions. True False
(d) For some $b$, the system $Ax = b$ has infinitely many solutions, and for some $b$, it is not solvable. True False
(e) For some $b$, the system $Ax = b$ is not solvable. True False

21. $A$, $B$ and $C$ are $3 \times 3$ matrices and $k$ is a scalar. Determine if each of the following statements is always true.

(a) If $A^2 = I_3$, then $A = I_3$ or $-I_3$. True False
(b) $\det(AB) = \det(A) \det(B)$. True False
(c) $\det(kA) = k^3 \det(A)$. True False
(d) If $C$ is invertible, the characteristic polynomial of $A$ is the same as that of $C^{-1}AC$. True False
(e) $(A - A^T)^T = A - A^T$. True False
(f) $\det(\text{adj}(A)) = \det(A)$. True False

22. For each of the following sets of vectors, determine if it is a vector (sub)space:

(a) The set of all vectors in $\mathbb{R}^4$ with the property $2x_1 + x_2 - 3x_3 + x_4 = 0$; Yes No

(b) The set of all vectors in $\mathbb{R}^4$ with the property $x_1^3 = x_2^3$, $x_3 = x_4$; Yes No
(c) The set of all vectors in \( \mathbb{R}^3 \), which have the form \((0, a - b + c, 3b + c)\) where \( a, b \) and \( c \) are arbitrary real numbers; Yes No

(d) The set of all polynomials \( P \) in the space of all polynomials of degree at most 7, with the property \( P(2) = 0 \); Yes No

(e) The set of all nonsingular matrices in the space of \( 3 \times 3 \) matrices; Yes No

23. Determine if each of the following sets of vectors is linearly independent or linearly dependent:

(a) \[
\begin{bmatrix}
2 & 3 & 4 \\
1 & 0 & 1 \\
3 & 1 & 1 \\
\end{bmatrix}
\] Independent Dependent

(b) \[
\begin{bmatrix}
1 & 1 & 0 \\
0 & 1 & 4 \\
0 & 1 & 5 \\
\end{bmatrix}
\] Independent Dependent

(c) \[
\begin{bmatrix}
0 & 1 & 0 \\
0 & 1 & 4 \\
\end{bmatrix}
\] Independent Dependent

(d) \[
\begin{bmatrix}
1 & 2 & 0 & [-3] \\
2 & 1 & 1 & 2 \\
5 & 7 & 1 & [-1] \\
\end{bmatrix}
\] Independent Dependent

(e) \[
\begin{bmatrix}
1 & 1 \\
2 & 0 \\
-1 & 1 \\
\end{bmatrix}
\] Independent Dependent

24. We have a \( 3 \times 3 \) matrix \( A = \begin{bmatrix} a & 1 & 2 \\ b & 3 & 4 \\ c & 5 & 6 \end{bmatrix} \) with \( \text{det}(A) = 3 \). Compute the determinant of the following matrix

(a) \[
\begin{bmatrix}
a - 2 & 1 & 2 \\
b - 4 & 3 & 4 \\
c - 6 & 5 & 6 \\
\end{bmatrix}
\]

(b) \[
\begin{bmatrix}
7a & 7 & 14 \\
b & 3 & 4 \\
c & 5 & 6 \\
\end{bmatrix}
\]

(c) \[
\begin{bmatrix}
a & 1 & 1 \\
b & 3 & 3 \\
c & 5 & 5 \\
\end{bmatrix}
\]
(d) $A^T$
(e) $(5A)^{-1}$

25. Let $A = \begin{bmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{bmatrix}$.

Determine if each of the following statement is true or false.

(a) If any two of $a$, $b$, $c$ have the same value, then $\det(A) = 0$. True False
(b) If $a$, $b$, $c$ have distinct values, then $A$ is nonsingular. True False
(c) If $a > b > c$, then $\det(A) < 0$. True False
(d) If $\text{rank}(A) = 1$, then $a$, $b$ and $c$ must all be equal. True False
(e) The rank of $A$ can never be equal to 2. True False

26. Let $A = \begin{bmatrix} 3 & -5 \\ 1 & -3 \end{bmatrix}$.

(a) Find all eigenvalues of $A$.
(b) For each eigenvalue above, find an eigenvector of $A$ associated to it.
(c) Find a diagonal matrix $D$ and a nonsingular matrix $P$ such that $P^{-1}AP = D$.
(d) Find $A^{47}$.

27. Let $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$.

(a) Find all eigenvalues of $A$.
(b) For each eigenvalue as above, find an eigenvector of $A$ associated to it.
(c) Find a diagonal matrix $D$ and an orthogonal matrix $P$ such that $P^TAP = D$.

28. Determine an invertible matrix $\hat{A}$ and a vector $\hat{b}$ such that the solution to $\hat{A}\hat{X} = \hat{b}$ is the least squares solution to $AX = b$, where

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ 1 & 0 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$  

29. Find the least squares fit line for the points $(1, 2), (2, 1), (3, 3), (4, 3)$.  

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30. Consider the homogeneous linear system of differential equations $\mathbf{x}'(t) = A\mathbf{x}$, where

$$A = \begin{bmatrix} 2 & -1 \\ 8 & 8 \end{bmatrix}.$$  

(a) Find all eigenvalues of $A$.

(b) Find the general solution to the linear system of differential equations.

(c) Solve the initial value problem for the given conditions $x_1(0) = 3$ and $x_2(0) = -10$.

31. Consider the homogeneous linear system of differential equations $\mathbf{x}'(t) = A\mathbf{x}$, where

$$A = \begin{bmatrix} 1 & -2 \\ 2 & -1 \end{bmatrix}.$$  

(a) Find all eigenvalues of $A$.

(b) Find the real-valued general solution to the linear system of differential equations.

(c) Solve the initial value problem for the given conditions $x_1(0) = 5$ and $x_2(0) = 1$.

32. Consider the homogeneous linear system of differential equations $\mathbf{x}'(t) = A\mathbf{x}$, where

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{bmatrix}.$$  

(a) Find all eigenvalues of $A$.

(b) Find the general solution to the linear system of differential equations.

(c) Solve the initial value problem for the given conditions

$$x_1(0) = -2, \ x_2(0) = 1, \ x_3(0) = 2$$