1.2 Solutions of Some Differential Equations

In each of Problems 19 through 26 draw a direction field for the given differential equation. Based on the direction field, determine the behavior of \( y \) as \( t \to \infty \). If this behavior depends on the initial value of \( y \) at \( t = 0 \), describe this dependency. Note that the right sides of these equations depend on \( t \) as well as \( y \); therefore their solutions can exhibit more complicated behavior than those in the text.

\[
\begin{align*}
19. \quad & y' = -2 + t - y \\
21. \quad & y' = e^{-t} + y \\
23. \quad & y' = 3 \sin t + 1 + y \\
25. \quad & y' = -(2t + y)/2y \\
20. \quad & y' = te^{-2t} - 2y \\
22. \quad & y' = t + 2y \\
24. \quad & y' = 2t - 1 - y^2 \\
26. \quad & y' = y^3/6 - y - t^2/3
\end{align*}
\]

1.2 Solutions of Some Differential Equations

In the preceding section we derived differential equations,

\[
\begin{align*}
\frac{m}{dt} &= mg - \gamma v \\
\frac{dp}{dt} &= rp - k
\end{align*}
\]

which model a falling object and a population of field mice preyed upon by owls, respectively. Both these equations are of the general form

\[
\frac{dy}{dt} = ay - b,
\]

where \( a \) and \( b \) are given constants. We were able to draw some important qualitative conclusions about the behavior of solutions of Eqs. (1) and (2) by considering the associated direction fields. To answer questions of a quantitative nature, however, we need to find the solutions themselves, and we now investigate how to do that.

Consider the equation

\[
\frac{dp}{dt} = 0.5 p - 450,
\]

which describes the interaction of certain populations of field mice and owls [see Eq. (8) of Section 1.1]. Find solutions of this equation.

To solve Eq. (4) we need to find functions \( p(t) \) that, when substituted into the equation, reduce it to an obvious identity. Here is one way to proceed. First, rewrite Eq. (4) in the form

\[
\frac{dp}{dt} = \frac{p - 900}{2},
\]

or, if \( p \neq 900 \),

\[
\frac{dp}{dt} = \frac{1}{2}. 
\]
Since, by the chain rule, the left side of Eq. (6) is the derivative of $\ln |p - 900|$ with respect to $t$, it follows that

$$\frac{d}{dt} \ln |p - 900| = \frac{1}{2}.$$  

(7)

Then, by integrating both sides of Eq. (7), we obtain

$$\ln |p - 900| = \frac{t}{2} + C,$$

(8)

where $C$ is an arbitrary constant of integration. Therefore, by taking the exponential of both sides of Eq. (8), we find that

$$|p - 900| = e^{C}e^{t/2},$$

(9)

or

$$p - 900 = \pm e^{C}e^{t/2},$$

(10)

and finally

$$p = 900 + ce^{t/2},$$

(11)

where $c = \pm e^{C}$ is also an arbitrary (nonzero) constant. Note that the constant function $p = 900$ is also a solution of Eq. (5) and that it is contained in the expression (11) if we allow $c$ to take the value zero. Graphs of Eq. (11) for several values of $c$ are shown in Figure 1.2.1.

![Graph of Eq. (11) for several values of $c$.](image)

FIGURE 1.2.1 Graphs of Eq. (11) for several values of $c$.

Note that they have the character inferred from the direction field in Figure 1.1.4. For instance, solutions lying on either side of the equilibrium solution $p = 900$ tend to diverge from that solution.

In Example 1 we found infinitely many solutions of the differential equation (4), corresponding to the infinitely many values that the arbitrary constant $c$ in Eq. (11)
might have. This is typical of what happens when you solve a differential equation. The solution process involves an integration, which brings with it an arbitrary constant, whose possible values generate an infinite family of solutions.

Frequently, we want to focus our attention on a single member of the infinite family of solutions by specifying the value of the arbitrary constant. Most often, we do this indirectly by specifying instead a point that must lie on the graph of the solution. For example, to determine the constant $c$ in Eq. (11), we could require that the population have a given value at a certain time, such as the value 850 at time $t = 0$. In other words, the graph of the solution must pass through the point $(0, 850)$. Symbolically, we can express this condition as

$$ p(0) = 850. \quad (12) $$

Then, substituting $t = 0$ and $p = 850$ into Eq. (11), we obtain

$$ 850 = 900 + c. $$

Hence $c = -50$, and by inserting this value in Eq. (11), we obtain the desired solution, namely,

$$ p = 900 - 50e^{t/2}. \quad (13) $$

The additional condition (12) that we used to determine $c$ is an example of an initial condition. The differential equation (4) together with the initial condition (12) form an initial value problem.

Now consider the more general problem consisting of the differential equation (3)

$$ \frac{dy}{dt} = ay - b $$

and the initial condition

$$ y(0) = y_0. \quad (14) $$

where $y_0$ is an arbitrary initial value. We can solve this problem by the same method as in Example 1. If $a \neq 0$ and $y \neq b/a$, then we can rewrite Eq. (3) as

$$ \frac{dy/dt}{y - (b/a)} = a. \quad (15) $$

By integrating both sides, we find that

$$ \ln|y - (b/a)| = at + C, \quad (16) $$

where $C$ is arbitrary. Then, taking the exponential of both sides of Eq. (16) and solving for $y$, we obtain

$$ y = (b/a) + ce^{at}, \quad (17) $$

where $c = \pm e^C$ is also arbitrary. Observe that $c = 0$ corresponds to the equilibrium solution $y = b/a$. Finally, the initial condition (14) requires that $c = y_0 - (b/a)$, so the solution of the initial value problem (3), (14) is

$$ y = (b/a) + [y_0 - (b/a)]e^{at}. \quad (18) $$

The expression (17) contains all possible solutions of Eq. (3) and is called the general solution. The geometrical representation of the general solution (17) is an infinite family of curves, called integral curves. Each integral curve is associated with
a particular value of \( c \), and is the graph of the solution corresponding to that value of \( c \). Satisfying an initial condition amounts to identifying the integral curve that passes through the given initial point.

To relate the solution (18) to Eq. (2), which models the field mouse population, we need only replace \( a \) by the growth rate \( r \) and \( b \) by the predation rate \( k \). Then the solution (18) becomes

\[
p = \frac{(k}{r}) + [p_0 - (k/r)]e^{rt},
\]

where \( p_0 \) is the initial population of field mice. The solution (19) confirms the conclusions reached on the basis of the direction field and Example 1. If \( p_0 = k/r \), then from Eq. (19) it follows that \( p = k/r \) for all \( t \); this is the constant, or equilibrium, solution. If \( p_0 \neq k/r \), then the behavior of the solution depends on the sign of the coefficient \( p_0 - (k/r) \) of the exponential term in Eq. (19). If \( p_0 > k/r \), then \( p \) grows exponentially with time \( t \); if \( p_0 < k/r \), then \( p \) decreases and eventually becomes zero, corresponding to extinction of the field mouse population. Negative values of \( p \), while possible for the expression (19), make no sense in the context of this particular problem.

To put the falling object equation (1) in the form (3), we must identify \( a \) with \(-\gamma/m\) and \( b \) with \(-g\). Making these substitutions in the solution (18), we obtain

\[
v = (mg/\gamma) + [v_0 - (mg/\gamma)]e^{-\gamma t/m},
\]

where \( v_0 \) is the initial velocity. Again, this solution confirms the conclusions reached in Section 1.1 on the basis of a direction field. There is an equilibrium, or constant, solution \( v = mg/\gamma \), and all other solutions tend to approach this equilibrium solution. The speed of convergence to the equilibrium solution is determined by the exponent \(-\gamma/m\). Thus, for a given mass \( m \) the velocity approaches the equilibrium value faster as the drag coefficient \( \gamma \) increases.

**Example 2**

A Falling Object (continued)

Suppose that, as in Example 2 of Section 1.1, we consider a falling object of mass \( m = 10 \text{ kg} \) and drag coefficient \( \gamma = 2 \text{ kg/sec} \). Then the equation of motion (1) becomes

\[
\frac{dv}{dt} = 9.8 - \frac{v}{5}.
\]

Suppose this object is dropped from a height of 300 m. Find its velocity at any time \( t \). How long will it take to fall to the ground, and how fast will it be moving at the time of impact?

The first step is to state an appropriate initial condition for Eq. (21). The word “dropped” in the statement of the problem suggests that the initial velocity is zero, so we will use the initial condition

\[
v(0) = 0.
\]

The solution of Eq. (21) can be found by substituting the values of the coefficients into the solution (20), but we will proceed instead to solve Eq. (21) directly. First, rewrite the equation as

\[
\frac{dv}{v - 49} = -\frac{1}{5}.\]
By integrating both sides we obtain
\[ \ln |v - 49| = -\frac{t}{5} + C, \]  
(24)

and then the general solution of Eq. (21) is
\[ v = 49 + ce^{-t/5}, \]  
(25)

where \( c \) is arbitrary. To determine \( c \), we substitute \( t = 0 \) and \( v = 0 \) from the initial condition (22) into Eq. (25), with the result that \( c = -49 \). Then the solution of the initial value problem (21), (22) is
\[ v = 49(1 - e^{-t/5}). \]  
(26)

Equation (26) gives the velocity of the falling object at any positive time (before it hits the ground, of course).

Graphs of the solution (25) for several values of \( c \) are shown in Figure 1.2.2, with the solution (26) shown by the heavy curve. It is evident that all solutions tend to approach the equilibrium solution \( v = 49 \). This confirms the conclusions we reached in Section 1.1.1 on the basis of the direction fields in Figures 1.1.2 and 1.1.3.

![Figure 1.2.2](image)

**FIGURE 1.2.2** Graphs of the solution (25) for several values of \( c \).

To find the velocity of the object when it hits the ground, we need to know the time at which impact occurs. In other words, we need to determine how long it takes the object to fall 300 m. To do this, we note that the distance \( x \) the object has fallen is related to its velocity \( v \) by the equation \( v = dx/dt \), or
\[ \frac{dx}{dt} = 49(1 - e^{-t/5}). \]  
(27)

Consequently,
\[ x = 49t + 245e^{-t/5} + c, \]  
(28)
where $c$ is an arbitrary constant of integration. The object starts to fall when $t = 0$, so we know that $x = 0$ when $t = 0$. From Eq. (28) it follows that $c = -245$, so the distance the object has fallen at time $t$ is given by

$$x = 49t + 245e^{-t/5} - 245.$$  \hspace{1cm} (29)

Let $T$ be the time at which the object hits the ground; then $x = 300$ when $t = T$. By substituting these values in Eq. (29) we obtain the equation

$$49T + 245e^{-T/5} - 545 = 0.$$  \hspace{1cm} (30)

The value of $T$ satisfying Eq. (30) can be readily approximated by a numerical process using a scientific calculator or computer, with the result that $T \approx 10.51$ sec. At this time, the corresponding velocity $v_T$ is found from Eq. (26) to be $v_T \approx 43.01$ m/sec.

**Further Remarks on Mathematical Modeling.** Up to this point we have related our discussion of differential equations to mathematical models of a falling object and of a hypothetical relation between field mice and owls. The derivation of these models may have been plausible, and possibly even convincing, but you should remember that the ultimate test of any mathematical model is whether its predictions agree with observations or experimental results. We have no actual observations or experimental results to use for comparison purposes here, but there are several sources of possible discrepancies.

In the case of the falling object the underlying physical principle (Newton’s law of motion) is well-established and widely applicable. However, the assumption that the drag force is proportional to the velocity is less certain. Even if this assumption is correct, the determination of the drag coefficient $\gamma$ by direct measurement presents difficulties. Indeed, sometimes one finds the drag coefficient indirectly, for example, by measuring the time of fall from a given height, and then calculating the value of $\gamma$ that predicts this time.

The model of the field mouse population is subject to various uncertainties. The determination of the growth rate $r$ and the predation rate $k$ depends on observations of actual populations, which may be subject to considerable variation. The assumption that $r$ and $k$ are constants may also be questionable. For example, a constant predation rate becomes harder to sustain as the population becomes smaller. Further, the model predicts that a population above the equilibrium value will grow exponentially larger and larger. This seems at variance with the behavior of actual populations; see the further discussion of population dynamics in Section 2.5.

Even if a mathematical model is incomplete or somewhat inaccurate, it may nevertheless be useful in explaining qualitative features of the problem under investigation. It may also be valid under some circumstances but not others. Thus you should always use good judgment and common sense in constructing mathematical models and in using their predictions.

**PROBLEMS**

1. Solve each of the following initial value problems and plot the solutions for several values of $y_0$. Then describe in a few words how the solutions resemble, and differ from, each other.
   (a) $dy/dt = -y + 5, \quad y(0) = y_0$
   (b) $dy/dt = -2y + 5, \quad y(0) = y_0$
   (c) $dy/dt = -2y + 10, \quad y(0) = y_0$
2. Follow the instructions for Problem 1 for the following initial value problems:
   (a) \( \frac{dy}{dt} = y - 5 \), \( y(0) = y_0 \)
   (b) \( \frac{dy}{dt} = 2y - 5 \), \( y(0) = y_0 \)
   (c) \( \frac{dy}{dt} = 2y - 10 \), \( y(0) = y_0 \)

3. Consider the differential equation
   \[ \frac{dy}{dt} = -ay + b, \]
   where both \( a \) and \( b \) are positive numbers.
   (a) Solve the differential equation.
   (b) Sketch the solution for several different initial conditions.
   (c) Describe how the solutions change under each of the following conditions:
      i. \( a \) increases.
      ii. \( b \) increases.
      iii. Both \( a \) and \( b \) increase, but the ratio \( b/a \) remains the same.

4. Here is an alternative way to solve the equation
   \[ \frac{dy}{dt} = ay - b. \]  
   (i)
   (a) Solve the simpler equation
       \[ \frac{dy}{dt} = ay. \]  
       (ii)
   Call the solution \( y_1(t) \).
   (b) Observe that the only difference between Eqs. (i) and (ii) is the constant \(-b\) in Eq. (i).
       Therefore it may seem reasonable to assume that the solutions of these two equations also differ only by a constant.
       Test this assumption by trying to find a constant \( k \) so that \( y = y_1(t) + k \) is a solution of Eq. (i).
   (c) Compare your solution from part (b) with the solution given in the text in Eq. (17).
       Note: This method can also be used in some cases in which the constant \( b \) is replaced by a function \( g(t) \).
       It depends on whether you can guess the general form that the solution is likely to take.
       This method is described in detail in Section 3.6 in connection with second order equations.

5. Use the method of Problem 4 to solve the equation
   \[ \frac{dy}{dt} = -ay + b. \]

6. The field mouse population in Example 1 satisfies the differential equation
   \[ \frac{dp}{dt} = 0.5p - 450. \]
   (a) Find the time at which the population becomes extinct if \( p(0) = 850 \).
   (b) Find the time of extinction if \( p(0) = p_0 \), where \( 0 < p_0 < 900 \).
   (c) Find the initial population \( p_0 \) if the population is to become extinct in 1 year.

7. Consider a population \( p \) of field mice that grows at a rate proportional to the current population,
   so that \( \frac{dp}{dt} = rp \).
   (a) Find the rate constant \( r \) if the population doubles in 30 days.
   (b) Find \( r \) if the population doubles in \( N \) days.

8. The falling object in Example 2 satisfies the initial value problem
   \[ \frac{dv}{dt} = 9.8 - \frac{v}{5}, \quad v(0) = 0. \]
   (a) Find the time that must elapse for the object to reach 98% of its limiting velocity.
   (b) How far does the object fall in the time found in part (a)?

9. Modify Example 2 so that the falling object experiences no air resistance.
   (a) Write down the modified initial value problem.
   (b) Determine how long it takes the object to reach the ground.
   (c) Determine its velocity at the time of impact.
10. A radioactive material, such as the isotope thorium-234, disintegrates at a rate proportional to the amount currently present. If \( Q(t) \) is the amount present at time \( t \), then \( \frac{dQ}{dt} = -rQ \), where \( r > 0 \) is the decay rate.

(a) If 100 mg of thorium-234 decays to 82.04 mg in 1 week, determine the decay rate \( r \).

(b) Find an expression for the amount of thorium-234 present at any time \( t \).

(c) Find the time required for the thorium-234 to decay to one-half its original amount.

11. The **half-life** of a radioactive material is the time required for an amount of this material to decay to one-half its original value. Show that, for any radioactive material that decays according to the equation \( Q' = -rQ \), the half-life \( \tau \) and the decay rate \( r \) satisfy the equation \( r \tau = \ln 2 \).

12. Radium-226 has a half-life of 1620 years. Find the time period during which a given amount of this material is reduced by one-quarter.

13. Consider an electric circuit containing a capacitor, resistor, and battery; see Figure 1.2.3. The charge \( Q(t) \) on the capacitor satisfies the equation \(^3\)

\[
R \frac{dQ}{dt} + \frac{Q}{C} = V,
\]

where \( R \) is the resistance, \( C \) is the capacitance, and \( V \) is the constant voltage supplied by the battery.

\[
\text{FIGURE 1.2.3} \quad \text{The electric circuit of Problem 13.}
\]

(a) If \( Q(0) = 0 \), find \( Q(t) \) at any time \( t \), and sketch the graph of \( Q \) versus \( t \).

(b) Find the limiting value \( Q_\infty \), that \( Q(t) \) approaches after a long time.

(c) Suppose that \( Q(t_1) = Q_\infty \) and that the battery is removed from the circuit at \( t = t_1 \). Find \( Q(t) \) for \( t > t_1 \) and sketch its graph.

14. A pond containing 1,000,000 gal of water is initially free of a certain undesirable chemical (see Problem 15 of Section 1.1). Water containing 0.01 g/gal of the chemical flows into the pond at a rate of 300 gal/hr and water also flows out of the pond at the same rate. Assume that the chemical is uniformly distributed throughout the pond.

(a) Let \( Q(t) \) be the amount of the chemical in the pond at time \( t \). Write down an initial value problem for \( Q(t) \).

(b) Solve the problem in part (a) for \( Q(t) \). How much chemical is in the pond after 1 year?

(c) At the end of 1 year the source of the chemical in the pond is removed and thereafter pure water flows into the pond and the mixture flows out at the same rate as before. Write down the initial value problem that describes this new situation.

(d) Solve the initial value problem in part (c). How much chemical remains in the pond after 1 additional year (2 years from the beginning of the problem)?

(e) How long does it take for \( Q(t) \) to be reduced to 10 g?

(f) Plot \( Q(t) \) versus \( t \) for 3 years.

15. Your swimming pool containing 60,000 gal of water has been contaminated by 5 kg of a nontoxic dye that leaves a swimmer’s skin an unattractive green. The pool’s filtering

\(^3\text{This equation results from Kirchhoff's laws, which are discussed later in Section 3.8.}\)
system can take water from the pool, remove the dye, and return the water to the pool at a rate of 200 gal/min.
(a) Write down the initial value problem for the filtering process; let \( q(t) \) be the amount of dye in the pool at any time \( t \).
(b) Solve the problem in part (a).
(c) You have invited several dozen friends to a pool party that is scheduled to begin in 4 hr. You have also determined that the effect of the dye is imperceptible if its concentration is less than 0.02 g/gal. Is your filtering system capable of reducing the dye concentration to this level within 4 hr?
(d) Find the time \( T \) at which the concentration of dye first reaches the value 0.02 g/gal.
(e) Find the flow rate that is sufficient to achieve the concentration 0.02 g/gal within 4 hr.

1.3 Classification of Differential Equations

The main purpose of this book is to discuss some of the properties of solutions of differential equations, and to describe some of the methods that have proved effective in finding solutions, or in some cases approximating them. To provide a framework for our presentation we describe here several useful ways of classifying differential equations.

**Ordinary and Partial Differential Equations.** One of the more obvious classifications is based on whether the unknown function depends on a single independent variable or on several independent variables. In the first case, only ordinary derivatives appear in the differential equation, and it is said to be an **ordinary differential equation**. In the second case, the derivatives are partial derivatives, and the equation is called a **partial differential equation**.

All the differential equations discussed in the preceding two sections are ordinary differential equations. Another example of an ordinary differential equation is

\[
L \frac{d^2 Q(t)}{dt^2} + R \frac{dQ(t)}{dt} + \frac{1}{C} Q(t) = E(t),
\]

for the charge \( Q(t) \) on a capacitor in a circuit with capacitance \( C \), resistance \( R \), and inductance \( L \); this equation is derived in Section 3.8. Typical examples of partial differential equations are the heat conduction equation

\[
a^2 \frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial u(x,t)}{\partial t},
\]

and the wave equation

\[
a^2 \frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial^2 u(x,t)}{\partial t^2}.
\]

Here, \( a^2 \) and \( a^2 \) are certain physical constants. The heat conduction equation describes the conduction of heat in a solid body and the wave equation arises in a variety of problems involving wave motion in solids or fluids. Note that in both Eqs. (2) and (3) the dependent variable \( u \) depends on the two independent variables \( x \) and \( t \).
Systems of Differential Equations. Another classification of differential equations depends on the number of unknown functions that are involved. If there is a single function to be determined, then one equation is sufficient. However, if there are two or more unknown functions, then a system of equations is required. For example, the Lotka–Volterra, or predator–prey, equations are important in ecological modeling. They have the form

\[
\begin{align*}
\frac{dx}{dt} &= ax - \alpha xy \\
\frac{dy}{dt} &= -cy + \gamma xy,
\end{align*}
\]

where \(x(t)\) and \(y(t)\) are the respective populations of the prey and predator species. The constants \(a, \alpha, c,\) and \(\gamma\) are based on empirical observations and depend on the particular species being studied. Systems of equations are discussed in Chapters 7 and 9; in particular, the Lotka–Volterra equations are examined in Section 9.5. It is not unusual in some areas of application to encounter systems containing a large number of equations.

Order. The order of a differential equation is the order of the highest derivative that appears in the equation. The equations in the preceding sections are all first order equations, while Eq. (1) is a second order equation. Equations (2) and (3) are second order partial differential equations. More generally, the equation

\[
F[t, u(t), u'(t), \ldots, u^{(n)}(t)] = 0
\]

is an ordinary differential equation of the \(n\)th order. Equation (5) expresses a relation between the independent variable \(t\) and the values of the function \(u\) and its first \(n\) derivatives \(u', u'', \ldots, u^{(n)}\). It is convenient and customary in differential equations to write \(y\) for \(u(t)\), with \(y', y'', \ldots, y^{(n)}\) standing for \(u'(t), u''(t), \ldots, u^{(n)}(t)\). Thus Eq. (5) is written as

\[
F(t, y, y', \ldots, y^{(n)}) = 0.
\]

For example,

\[
y'' + 2ey' + yy' = t^4
\]

is a third order differential equation for \(y = u(t)\). Occasionally, other letters will be used instead of \(t\) and \(y\) for the independent and dependent variables; the meaning should be clear from the context.

We assume that it is always possible to solve a given ordinary differential equation for the highest derivative, obtaining

\[
y^{(n)} = f(t, y, y', y'', \ldots, y^{(n-1)}).
\]

We study only equations of the form (8). This is mainly to avoid the ambiguity that may arise because a single equation of the form (6) may correspond to several equations of the form (8). For example, the equation

\[
y'^2 + ty' + 4y = 0
\]

leads to the two equations

\[
y' = \frac{-t + \sqrt{t^2 - 16y}}{2} \quad \text{or} \quad y' = \frac{-t - \sqrt{t^2 - 16y}}{2}.
\]
1.3 Classification of Differential Equations

Linear and Nonlinear Equations. A crucial classification of differential equations is whether they are linear or nonlinear. The ordinary differential equation

\[ F(t, y, y', \ldots, y^{(n)}) = 0 \]

is said to be linear if \( F \) is a linear function of the variables \( y, y', \ldots, y^{(n)} \); a similar definition applies to partial differential equations. Thus the general linear ordinary differential equation of order \( n \) is

\[ a_0(t)y^{(n)} + a_1(t)y^{(n-1)} + \cdots + a_n(t)y = g(t). \quad (11) \]

Most of the equations you have seen thus far in this book are linear; examples are the equations in Sections 1.1 and 1.2 describing the falling object and the field mouse population. Similarly, in this section, Eq. (1) is a linear ordinary differential equation and Eqs. (2) and (3) are linear partial differential equations. An equation that is not of the form (11) is a nonlinear equation. Equation (7) is nonlinear because of the term \( yy' \). Similarly, each equation in the system (4) is nonlinear because of the terms that involve the product \( xy \).

A simple physical problem that leads to a nonlinear differential equation is the oscillating pendulum. The angle \( \theta \) that an oscillating pendulum of length \( L \) makes with the vertical direction (see Figure 1.3.1) satisfies the equation

\[ \frac{d^2\theta}{dt^2} + \frac{g}{L} \sin \theta = 0, \quad (12) \]

whose derivation is outlined in Problem 29. The presence of the term involving \( \sin \theta \) makes Eq. (12) nonlinear.

The mathematical theory and methods for solving linear equations are highly developed. In contrast, for nonlinear equations the theory is more complicated and methods of solution are less satisfactory. In view of this, it is fortunate that many significant problems lead to linear ordinary differential equations or can be approximated by linear equations. For example, for the pendulum, if the angle \( \theta \) is small, then \( \sin \theta \approx \theta \) and Eq. (12) can be approximated by the linear equation

\[ \frac{d^2\theta}{dt^2} + \frac{g}{L} \theta = 0. \quad (13) \]

This process of approximating a nonlinear equation by a linear one is called linearization and it is an extremely valuable way to deal with nonlinear equations. Nevertheless, there are many physical phenomena that simply cannot be represented adequately.
by linear equations; to study these phenomena it is essential to deal with nonlinear equations.

In an elementary text it is natural to emphasize the simpler and more straightforward parts of the subject. Therefore the greater part of this book is devoted to linear equations and various methods for solving them. However, Chapters 8 and 9, as well as parts of Chapter 2, are concerned with nonlinear equations. Whenever it is appropriate, we point out why nonlinear equations are, in general, more difficult, and why many of the techniques that are useful in solving linear equations cannot be applied to nonlinear equations.

**Solutions.** A solution of the ordinary differential equation (8) on the interval \( \alpha < t < \beta \) is a function \( \phi \) such that \( \phi', \phi'', \ldots, \phi^{(n)} \) exist and satisfy

\[
\phi^{(n)}(t) = f[t, \phi(t), \phi'(t), \ldots, \phi^{(n-1)}(t)]
\]  

for every \( t \) in \( \alpha < t < \beta \). Unless stated otherwise, we assume that the function \( f \) of Eq. (8) is a real-valued function, and we are interested in obtaining real-valued solutions \( y = \phi(t) \).

Recall that in Section 1.2 we found solutions of certain equations by a process of direct integration. For instance, we found that the equation

\[
\frac{dp}{dt} = 0.5p - 450
\]  

has the solution

\[
p = 900 + ce^{t/2},
\]  

where \( c \) is an arbitrary constant. It is often not so easy to find solutions of differential equations. However, if you find a function that you think may be a solution of a given equation, it is usually relatively easy to determine whether the function is actually a solution simply by substituting the function into the equation. For example, in this way it is easy to show that the function \( y_1(t) = \cos t \) is a solution of

\[
y'' + y = 0
\]  

for all \( t \). To confirm this, observe that \( y_1'(t) = -\sin t \) and \( y_1''(t) = -\cos t \); then it follows that \( y_1''(t) + y_1(t) = 0 \). In the same way you can easily show that \( y_2(t) = \sin t \) is also a solution of Eq. (17). Of course, this does not constitute a satisfactory way to solve most differential equations because there are far too many possible functions for you to have a good chance of finding the correct one by a random choice. Nevertheless, it is important to realize that you can verify whether any proposed solution is correct by substituting it into the differential equation. For a problem of any importance this can be a very useful check and is one that you should make a habit of considering.

**Some Important Questions.** Although for the equations (15) and (17) we are able to verify that certain simple functions are solutions, in general we do not have such solutions readily available. Thus a fundamental question is the following: Does an equation of the form (8) always have a solution? The answer is “No.” Merely writing down an equation of the form (8) does not necessarily mean that there is a function \( y = \phi(t) \) that satisfies it. So, how can we tell whether some particular equation has a solution? This is the question of existence of a solution, and it is answered by theorems stating that under certain restrictions on the function \( f \) in Eq. (8), the equation always
has solutions. However, this is not a purely mathematical concern, for at least two reasons. If a problem has no solution, we would prefer to know that fact before investing time and effort in a vain attempt to solve the problem. Further, if a sensible physical problem is modeled mathematically as a differential equation, then the equation should have a solution. If it does not, then presumably there is something wrong with the formulation. In this sense an engineer or scientist has some check on the validity of the mathematical model.

Second, if we assume that a given differential equation has at least one solution, the question arises as to how many solutions it has, and what additional conditions must be specified to single out a particular solution. This is the question of uniqueness. In general, solutions of differential equations contain one or more arbitrary constants of integration, as does the solution (16) of Eq. (15). Equation (16) represents an infinity of functions corresponding to the infinity of possible choices of the constant $c$. As we saw in Section 1.2, if $p$ is specified at some time $t$, this condition will determine a value for $c$; even so, we have not yet ruled out the possibility that there may be other solutions of Eq. (15) that also have the prescribed value of $p$ at the prescribed time $t$.

The issue of uniqueness also has practical implications. If we are fortunate enough to find a solution of a given problem, and if we know that the problem has a unique solution, then we can be sure that we have completely solved the problem. If there may be other solutions, then perhaps we should continue to search for them.

A third important question is: Given a differential equation of the form (8), can we actually determine a solution, and if so, how? Note that if we find a solution of the given equation, we have at the same time answered the question of the existence of a solution. However, without knowledge of existence theory we might, for example, use a computer to find a numerical approximation to a “solution” that does not exist. On the other hand, even though we may know that a solution exists, it may be that the solution is not expressible in terms of the usual elementary functions—polynomial, trigonometric, exponential, logarithmic, and hyperbolic functions. Unfortunately, this is the situation for most differential equations. Thus, while we discuss elementary methods that can be used to obtain solutions of certain relatively simple problems, it is also important to consider methods of a more general nature that can be applied to more difficult problems.

**Computer Use in Differential Equations.** A computer can be an extremely valuable tool in the study of differential equations. For many years computers have been used to execute numerical algorithms, such as those described in Chapter 8, to construct numerical approximations to solutions of differential equations. At the present time these algorithms have been refined to an extremely high level of generality and efficiency. A few lines of computer code, written in a high-level programming language and executed (often within a few seconds) on a relatively inexpensive computer, suffice to solve numerically a wide range of differential equations. More sophisticated routines are also readily available. These routines combine the ability to handle very large and complicated systems with numerous diagnostic features that alert the user to possible problems as they are encountered.

The usual output from a numerical algorithm is a table of numbers, listing selected values of the independent variable and the corresponding values of the dependent variable. With appropriate software it is easy to display the solution of a differential equation graphically, whether the solution has been obtained numerically or as the result of an analytical procedure of some kind. Such a graphical display is often much more
illuminating and helpful in understanding and interpreting the solution of a differential equation than a table of numbers or a complicated analytical formula. There are on the market several well-crafted and relatively inexpensive special-purpose software packages for the graphical investigation of differential equations. The widespread availability of personal computers has brought powerful computational and graphical capability within the reach of individual students. You should consider, in the light of your own circumstances, how best to take advantage of the available computing resources. You will surely find it enlightening to do so.

Another aspect of computer use that is very relevant to the study of differential equations is the availability of extremely powerful and general software packages that can perform a wide variety of mathematical operations. Among these are Maple®, Mathematica®, and MATLAB®, each of which can be used on various kinds of personal computers or workstations. All three of these packages can execute extensive numerical computations and have versatile graphical facilities. In addition, Maple and Mathematica also have very extensive analytical capabilities. For example, they can perform the analytical steps involved in solving many differential equations, often in response to a single command. Anyone who expects to deal with differential equations in more than a superficial way should become familiar with at least one of these products and explore the ways in which it can be used.

For you, the student, these computing resources have an effect on how you should study differential equations. To become confident in using differential equations, it is essential to understand how the solution methods work, and this understanding is achieved, in part, by working out a sufficient number of examples in detail. However, eventually you should plan to delegate as many as possible of the routine (often repetitive) details to a computer, while you focus more attention on the proper formulation of the problem and on the interpretation of the solution. Our viewpoint is that you should always try to use the best methods and tools available for each task. In particular, you should strive to combine numerical, graphical, and analytical methods so as to attain maximum understanding of the behavior of the solution and of the underlying process that the problem models. You should also remember that some tasks can best be done with pencil and paper, while others require a calculator or computer. Good judgment is often needed in selecting a judicious combination.

**PROBLEMS**

In each of Problems 1 through 6 determine the order of the given differential equation; also state whether the equation is linear or nonlinear.

1. \( t^2 \frac{d^2 y}{dt^2} + t \frac{dy}{dt} + 2y = \sin t \)
2. \( (1 + y^2) \frac{d^2 y}{dt^2} + t \frac{dy}{dt} + y = e^t \)
3. \( \frac{d^4 y}{dt^4} + \frac{d^3 y}{dt^3} + \frac{d^2 y}{dt^2} + \frac{dy}{dt} + y = 1 \)
4. \( \frac{dy}{dt} + ty^2 = 0 \)
5. \( \frac{d^2 y}{dt^2} + \sin(t + y) = \sin t \)
6. \( \frac{d^3 y}{dt^3} + t \frac{dy}{dt} + (\cos^2 t)y = t^3 \)

In each of Problems 7 through 14 verify that the given function or functions is a solution of the differential equation.

7. \( y'' - y = 0; \quad y_1(t) = e^t, \quad y_2(t) = \cosh t \)
8. \( y'' + 2y' - 3y = 0; \quad y_1(t) = e^{-3t}, \quad y_2(t) = e^t \)
9. \( ty' - y = t^2; \quad y = 3t + t^2 \)
10. \( y'''' + 4y''' + 3y = t; \quad y_1(t) = t/3, \quad y_2(t) = e^{-t} + t/3 \)
11. \( 2t^2y' + 3ty' - y = 0, \quad t > 0; \quad y_1(t) = t^{1/2}, \quad y_2(t) = t^{-1} \)
12. \( t^2y'' + 5ty' + 4y = 0, \quad t > 0; \quad y_1(t) = t^{-2}, \quad y_2(t) = t^{-2} \ln t \)
13. \( y'' + y = \sec t, \quad 0 < t < \pi/2; \quad y = (\cos t) \ln \cos t + t \sin t \)
14. \( y' - 2ty = 1; \quad y = e^t \int_0^t e^{-s^2} ds + e^2 \)

In each of Problems 15 through 18 determine the values of \( r \) for which the given differential equation has solutions of the form \( y = e^{rt} \).
15. \( y' + 2y = 0 \)
16. \( y'' - y = 0 \)
17. \( y'' + y' - 6y = 0 \)
18. \( y'' - 3y'' + 2y' = 0 \)

In each of Problems 19 and 20 determine the values of \( r \) for which the given differential equation has solutions of the form \( y = e^{rt} \) for \( t > 0 \).
19. \( t^2y'' + 4ty' + 2y = 0 \)
20. \( t^2y'' - 4ty' + 4y = 0 \)

In each of Problems 21 through 24 determine the order of the given partial differential equation; also state whether the equation is linear or nonlinear. Partial derivatives are denoted by subscripts.
21. \( u_{xx} + u_{xy} + u_{zz} = 0 \)
22. \( u_{xx} + u_{yy} + uu_x + uu_y + u = 0 \)
23. \( u_{xxx} + 2u_{xxy} + u_{xyy} = 0 \)
24. \( u_{t} + uu_{x} = 1 + u_{xx} \)

In each of Problems 25 through 28 verify that the given function or functions is a solution of the given partial differential equation.
25. \( u_{xx} + u_{yy} = 0; \quad u_1(x, y) = \cos x \cosh y, \quad u_2(x, y) = \ln(x^2 + y^2) \)
26. \( \frac{\partial^2 u}{\partial x^2} = u_t; \quad u_1(x, t) = e^{-at^2} \sin x, \quad u_2(x, t) = e^{-at^2} \sin \lambda x, \quad \lambda \text{ a real constant} \)
27. \( \frac{\partial^2 u}{\partial x^2} = u_t; \quad u_1(x, t) = \sin \lambda x \sin \lambda at, \quad u_2(x, t) = \sin(x - at), \quad \lambda \text{ a real constant} \)
28. \( \frac{\partial^2 u}{\partial x^2} = u_t; \quad u = (\pi/4)^{1/2} e^{-x^2/4a^2}, \quad t > 0 \)

29. Follow the steps indicated here to derive the equation of motion of a pendulum, Eq. (12) in the text. Assume that the rod is rigid and weightless, that the mass is a point mass, and that there is no friction or drag anywhere in the system.
   (a) Assume that the mass is in an arbitrary displaced position, indicated by the angle \( \theta \). Draw a free-body diagram showing the forces acting on the mass.
   (b) Apply Newton’s law of motion in the direction tangential to the circular arc on which the mass moves. Then the tensile force in the rod does not enter the equation. Observe that you need to find the component of the gravitational force in the tangential direction. Observe also that the linear acceleration, as opposed to the angular acceleration, is \( Ld^2\theta/dt^2 \), where \( L \) is the length of the rod.
   (c) Simplify the result from part (b) to obtain Eq. (12) of the text.

1.4 Historical Remarks

Without knowing something about differential equations and methods of solving them, it is difficult to appreciate the history of this important branch of mathematics. Further, the development of differential equations is intimately interwoven with the general development of mathematics and cannot be separated from it. Nevertheless, to provide some historical perspective, we indicate here some of the major trends in the history of