2.2 Separable Equations

In Sections 1.2 and 2.1 we used a process of direct integration to solve first order linear equations of the form

\[
\frac{dy}{dt} = ay + b,
\]  

(1)

where \(a\) and \(b\) are constants. We will now show that this process is actually applicable to a much larger class of equations.

We will use \(x\) to denote the independent variable in this section rather than \(t\) for two reasons. In the first place, different letters are frequently used for the variables in a differential equation, and you should not become too accustomed to using a single pair. In particular, \(x\) often occurs as the independent variable. Further, we want to reserve \(t\) for another purpose later in the section.

The general first order equation is

\[
\frac{dy}{dx} = f(x, y).
\]  

(2)

Linear equations were considered in the preceding section, but if Eq. (2) is nonlinear, then there is no universally applicable method for solving the equation. Here, we consider a subclass of first order equations for which a direct integration process can be used.

To identify this class of equations we first rewrite Eq. (2) in the form

\[
M(x, y) + N(x, y)\frac{dy}{dx} = 0.
\]  

(3)

It is always possible to do this by setting \(M(x, y) = -f(x, y)\) and \(N(x, y) = 1\), but there may be other ways as well. In the event that \(M\) is a function of \(x\) only and \(N\) is a function of \(y\) only, then Eq. (3) becomes

\[
M(x) + N(y)\frac{dy}{dx} = 0.
\]  

(4)

Such an equation is said to be separable, because if it is written in the differential form

\[
M(x)\, dx + N(y)\, dy = 0,
\]  

(5)

then, if you wish, terms involving each variable may be separated by the equals sign. The differential form (5) is also more symmetric and tends to diminish the distinction between independent and dependent variables.

Show that the equation

\[
\frac{dy}{dx} = \frac{x^2}{1 - y^2}
\]  

(6)

is separable, and then find an equation for its integral curves.

If we write Eq. (6) as

\[-x^2 + (1 - y^2)\frac{dy}{dx} = 0,
\]  

(7)
then it has the form (4) and is therefore separable. Next, observe that the first term in Eq. (7) is the derivative of $-x^3/3$ and that the second term, by means of the chain rule, is the derivative with respect to $x$ of $y - y^3/3$. Thus Eq. (7) can be written as

$$\frac{d}{dx} \left( -\frac{x^3}{3} \right) + \frac{d}{dx} \left( y - \frac{y^3}{3} \right) = 0,$$

or

$$\frac{d}{dx} \left( -\frac{x^3}{3} + y - \frac{y^3}{3} \right) = 0.$$

Therefore

$$-x^3 + 3y - y^3 = c,$$

(8)

where $c$ is an arbitrary constant, is an equation for the integral curves of Eq. (6). A direction field and several integral curves are shown in Figure 2.2.1. An equation of the integral curve passing through a particular point $(x_0, y_0)$ can be found by substituting $x_0$ and $y_0$ for $x$ and $y$, respectively, in Eq. (8) and determining the corresponding value of $c$. Any differentiable function $y = \phi(x)$ that satisfies Eq. (8) is a solution of Eq. (6).

![Direction field and integral curves of $y' = x^2/(1 - y^2)$.](image)

**FIGURE 2.2.1** Direction field and integral curves of $y' = x^2/(1 - y^2)$.

Essentially the same procedure can be followed for any separable equation. Returning to Eq. (4), let $H_1$ and $H_2$ be any functions such that

$$H_1'(x) = M(x), \quad H_2'(y) = N(y);$$

(9)
then Eq. (4) becomes

\[ H_1'(x) + H_2'(y) \frac{dy}{dx} = 0. \]  

(10)

According to the chain rule,

\[ H_2'(y) \frac{dy}{dx} = \frac{d}{dx} H_2(y). \]  

(11)

Consequently, Eq. (10) becomes

\[ \frac{d}{dx} [H_1(x) + H_2(y)] = 0. \]  

(12)

By integrating Eq. (12) we obtain

\[ H_1(x) + H_2(y) = c, \]  

(13)

where \( c \) is an arbitrary constant. Any differentiable function \( y = \phi(x) \) that satisfies Eq. (13) is a solution of Eq. (4); in other words, Eq. (13) defines the solution implicitly rather than explicitly. The functions \( H_1 \) and \( H_2 \) are any antiderivatives of \( M \) and \( N \), respectively. In practice, Eq. (13) is usually obtained from Eq. (5) by integrating the first term with respect to \( x \) and the second term with respect to \( y \).

If, in addition to the differential equation, an initial condition

\[ y(x_0) = y_0 \]  

(14)

is prescribed, then the solution of Eq. (4) satisfying this condition is obtained by setting \( x = x_0 \) and \( y = y_0 \) in Eq. (13). This gives

\[ c = H_1(x_0) + H_2(y_0). \]  

(15)

Substituting this value of \( c \) in Eq. (13) and noting that

\[ H_1(x) - H_1(x_0) = \int_{x_0}^{x} M(s) \, ds, \quad H_2(y) - H_2(y_0) = \int_{y_0}^{y} N(s) \, ds, \]

we obtain

\[ \int_{x_0}^{x} M(s) \, ds + \int_{y_0}^{y} N(s) \, ds = 0. \]  

(16)

Equation (16) is an implicit representation of the solution of the differential equation (4) that also satisfies the initial condition (14). You should bear in mind that the determination of an explicit formula for the solution requires that Eq. (16) be solved for \( y \) as a function of \( x \). Unfortunately, it is often impossible to do this analytically; in such cases one can resort to numerical methods to find approximate values of \( y \) for given values of \( x \).

**Example 2**

Solve the initial value problem

\[ \frac{dy}{dx} = \frac{3x^2 + 4x + 2}{2(y - 1)}, \quad y(0) = -1, \]  

(17)

and determine the interval in which the solution exists.
The differential equation can be written as
\[
2(y - 1) \, dy = (3x^2 + 4x + 2) \, dx.
\]
Integrating the left side with respect to \( y \) and the right side with respect to \( x \) gives
\[
y^2 - 2y = x^3 + 2x^2 + 2x + c,
\]
where \( c \) is an arbitrary constant. To determine the solution satisfying the prescribed initial condition, we substitute \( x = 0 \) and \( y = -1 \) in Eq. (18), obtaining \( c = 3 \). Hence the solution of the initial value problem is given implicitly by
\[
y^2 - 2y = x^3 + 2x^2 + 2x + 3.
\]
To obtain the solution explicitly we must solve Eq. (19) for \( y \) in terms of \( x \). This is a simple matter in this case, since Eq. (19) is quadratic in \( y \), and we obtain
\[
y = 1 \pm \sqrt{x^3 + 2x^2 + 2x + 4}.
\]
Equation (20) gives two solutions of the differential equation, only one of which, however, satisfies the given initial condition. This is the solution corresponding to the minus sign in Eq. (20), so that we finally obtain
\[
y = \phi(x) = 1 - \sqrt{x^3 + 2x^2 + 2x + 4}
\]
as the solution of the initial value problem (17). Note that if the plus sign is chosen by mistake in Eq. (20), then we obtain the solution of the same differential equation that satisfies the initial condition \( y(0) = 3 \). Finally, to determine the interval in which the solution (21) is valid, we must find the interval in which the quantity under the radical is positive. The only real zero of this expression is \( x = -2 \), so the desired interval is \( x > -2 \). The solution of the initial value problem and some other integral curves of the differential equation are shown in Figure 2.2.2. Observe that the boundary of the interval of validity of the solution (20) is determined by the point \((-2, 1)\) at which the tangent line is vertical.

FIGURE 2.2.2 Integral curves of \( y' = (3x^2 + 4x + 2)/2(y - 1) \).
Find the solution of the initial value problem
\[ \frac{dy}{dx} = \frac{y \cos x}{1 + 2y^2}, \quad y(0) = 1. \] (22)

Observe that \( y = 0 \) is a solution of the given differential equation. To find other solutions, assume that \( y \neq 0 \) and write the differential equation in the form
\[ \frac{1 + 2y^2}{y} \, dy = \cos x \, dx. \] (23)

Then, integrating the left side with respect to \( y \) and the right side with respect to \( x \), we obtain
\[ \ln |y| + y^2 = \sin x + c. \] (24)

To satisfy the initial condition we substitute \( x = 0 \) and \( y = 1 \) in Eq. (24); this gives \( c = 1 \). Hence the solution of the initial value problem (22) is given implicitly by
\[ \ln |y| + y^2 = \sin x + 1. \] (25)

Since Eq. (25) is not readily solved for \( y \) as a function of \( x \), further analysis of this problem becomes more delicate. One fairly evident fact is that no solution crosses the \( x \)-axis. To see this, observe that the left side of Eq. (25) becomes infinite if \( y = 0 \); however, the right side never becomes unbounded, so no point on the \( x \)-axis satisfies Eq. (25). Thus, for the solution of Eqs. (22) it follows that \( y > 0 \) always. Consequently, the absolute value bars in Eq. (25) can be dropped. It can also be shown that the interval of definition of the solution of the initial value problem (22) is the entire \( x \)-axis. Some integral curves of the given differential equation, including the solution of the initial value problem (22), are shown in Figure 2.2.3.

![Figure 2.2.3](image)

**FIGURE 2.2.3** Integral curves of \( y' = (y \cos x)/(1 + 2y^2) \).

The investigation of a first order nonlinear equation can sometimes be facilitated by regarding both \( x \) and \( y \) as functions of a third variable \( t \). Then
\[ \frac{dy}{dx} = \frac{dy/dt}{dx/dt}. \] (26)
If the differential equation is
\[ \frac{dy}{dx} = \frac{F(x, y)}{G(x, y)}, \]  
(27)
then, by comparing numerators and denominators in Eqs. (26) and (27), we obtain the system
\[ \frac{dx}{dt} = G(x, y), \quad \frac{dy}{dt} = F(x, y). \]  
(28)
At first sight it may seem unlikely that a problem will be simplified by replacing a single equation by a pair of equations, but, in fact, the system (28) may well be more amenable to investigation than the single equation (27). Chapter 9 is devoted to nonlinear systems of the form (28).

Note: In Example 2 it was not difficult to solve explicitly for \( y \) as a function of \( x \) and to determine the exact interval in which the solution exists. However, this situation is exceptional, and often it will be better to leave the solution in implicit form, as in Examples 1 and 3. Thus, in the problems below and in other sections where nonlinear equations appear, the terminology “solve the following differential equation” means to find the solution explicitly if it is convenient to do so, but otherwise to find an implicit formula for the solution.

### PROBLEMS

In each of Problems 1 through 8 solve the given differential equation.

1. \( y' = \frac{x^2}{y} \)
2. \( y' = x^2/y(1+x^3) \)
3. \( y' + y^2 \sin x = 0 \)
4. \( y' = \frac{(3x^2 - 1)}{(3 + 2y)} \)
5. \( y' = (\cos^2 x)(\cos^2 2y) \)
6. \( xy' = (1 - y^3)^{1/2} \)
7. \[ \frac{dy}{dx} = \frac{x - e^{-x}}{y + e^x} \]
8. \[ \frac{dy}{dx} = \frac{x^2}{1 + y^2} \]

In each of Problems 9 through 20:
(a) Find the solution of the given initial value problem in explicit form.
(b) Plot the graph of the solution.
(c) Determine (at least approximately) the interval in which the solution is defined.

9. \( y' = (1 - 2x)^2, \quad y(0) = -1/6 \)
10. \( y' = (1 - 2x)/y, \quad y(1) = -2 \)
11. \( x \frac{dx}{dt} + ye^{-x} \frac{dy}{dt} = 0, \quad y(0) = 1 \)
12. \( \frac{dr}{d\theta} = r^2/\theta, \quad r(1) = 2 \)
13. \( y' = 2x/(y + x^2), \quad y(0) = -2 \)
14. \( y' = xy^3(1 + x^2)^{-1/2}, \quad y(0) = 1 \)
15. \( y' = 2x/(1 + 2y), \quad y(2) = 0 \)
16. \( y' = x(x^2 + 1)/4y^3, \quad y(0) = -1/\sqrt{2} \)
17. \( y' = (3x^2 - e^x)/(2y - 5), \quad y(0) = 1 \)
18. \( y' = (e^{-x} - e^x)/(3 + 4y), \quad y(0) = 1 \)
19. \( \sin 2x \frac{dx}{dt} + \cos 3y \frac{dy}{dt} = 0, \quad y(\pi/2) = \pi/3 \)
20. \( y^2(1 - x^2)^{1/2} \frac{dy}{dx} = \arcsin x \frac{dx}{dx}, \quad y(0) = 0 \)

Some of the results requested in Problems 21 through 28 can be obtained either by solving the given equations analytically, or by plotting numerically generated approximations to the solutions. Try to form an opinion as to the advantages and disadvantages of each approach.

21. Solve the initial value problem
\[ y' = (1 + 3x^2)/(3y^2 - 6y), \quad y(0) = 1 \]
and determine the interval in which the solution is valid.

**Hint:** To find the interval of definition, look for points where the integral curve has a vertical tangent.
22. Solve the initial value problem
\[ y' = \frac{3x^2}{(3y^2 - 4)}, \quad y(1) = 0 \]
and determine the interval in which the solution is valid.
*Hint:* To find the interval of definition, look for points where the integral curve has a vertical tangent.

23. Solve the initial value problem
\[ y' = 2y^2 + xy^2, \quad y(0) = 1 \]
and determine where the solution attains its minimum value.

24. Solve the initial value problem
\[ y' = \frac{(2 - e^x)}{(3 + 2y)}, \quad y(0) = 0 \]
and determine where the solution attains its maximum value.

25. Solve the initial value problem
\[ y' = 2 \cos 2x/(3 + 2y), \quad y(0) = -1 \]
and determine where the solution attains its maximum value.

26. Solve the initial value problem
\[ y' = 2(1 + x)(1 + y^2), \quad y(0) = 0 \]
and determine where the solution attains its minimum value.

27. Consider the initial value problem
\[ y' = ty(4 - y)/3, \quad y(0) = y_0. \]
(a) Determine how the behavior of the solution as \( t \) increases depends on the initial value \( y_0 \).
(b) Suppose that \( y_0 = 0.5 \). Find the time \( T \) at which the solution first reaches the value 3.98.

28. Consider the initial value problem
\[ y' = ty(4 - y)/(1 + t), \quad y(0) = y_0 > 0. \]
(a) Determine how the solution behaves as \( t \to \infty \).
(b) If \( y_0 = 2 \), find the time \( T \) at which the solution first reaches the value 3.99.
(c) Find the range of initial values for which the solution lies in the interval \( 3.99 < y < 4.01 \) by the time \( t = 2 \).

29. Solve the equation
\[ \frac{dy}{dx} = \frac{ay + b}{cy + d}, \]
where \( a, b, c, \) and \( d \) are constants.

**Homogeneous Equations.** If the right side of the equation \( dy/dx = f(x, y) \) can be expressed as a function of the ratio \( y/x \) only, then the equation is said to be homogeneous. Such equations can always be transformed into separable equations by a change of the dependent variable. Problem 30 illustrates how to solve first order homogeneous equations.

30. Consider the equation
\[ \frac{dy}{dx} = \frac{y - 4x}{x - y}. \]
(a) Show that Eq. (i) can be rewritten as
\[ \frac{dy}{dx} = \frac{(y/x) - 4}{1 - (y/x)}; \]  
thus Eq. (i) is homogeneous.

(b) Introduce a new dependent variable \( v \) so that \( v = y/x \), or \( y = xv(x) \). Express \( dy/dx \) in terms of \( x, v, \) and \( dv/dx \).

(c) Replace \( y \) and \( dy/dx \) in Eq. (ii) by the expressions from part (b) that involve \( v \) and \( dv/dx \). Show that the resulting differential equation is
\[ v + x \frac{dv}{dx} = \frac{v - 4}{1 - v}, \]
or
\[ x \frac{dv}{dx} = \frac{v^2 - 4}{1 - v}. \]

Observe that Eq. (iii) is separable.

(d) Solve Eq. (iii) for \( v \) in terms of \( x \).

(e) Find the solution of Eq. (i) by replacing \( v \) by \( y/x \) in the solution in part (d).

(f) Draw a direction field and some integral curves for Eq. (i). Recall that the right side of Eq. (i) actually depends only on the ratio \( y/x \). This means that integral curves have the same slope at all points on any given straight line through the origin, although the slope changes from one line to another. Therefore the direction field and the integral curves are symmetric with respect to the origin. Is this symmetry property evident from your plot?

The method outlined in Problem 30 can be used for any homogeneous equation. That is, the substitution \( y = xv(x) \) transforms a homogeneous equation into a separable equation. The latter equation can be solved by direct integration, and then replacing \( v \) by \( y/x \) gives the solution to the original equation. In each of Problems 31 through 38:

(a) Show that the given equation is homogeneous.

(b) Solve the differential equation.

(c) Draw a direction field and some integral curves. Are they symmetric with respect to the origin?

\[ 31. \quad \frac{dy}{dx} = \frac{x^2 + xy + y^2}{x^2} \]
\[ 32. \quad \frac{dy}{dx} = \frac{x^2 + 3y^2}{2xy} \]
\[ 33. \quad \frac{dy}{dx} = \frac{4y - 3x}{2x - y} \]
\[ 34. \quad \frac{dy}{dx} = \frac{4x + 3y}{2x + y} \]
\[ 35. \quad \frac{dy}{dx} = \frac{x + 3y}{x - y} \]
\[ 36. \quad (x^2 + 3xy + y^2) \, dx - x^2 \, dy = 0 \]
\[ 37. \quad \frac{dy}{dx} = \frac{x^2 - 3y^2}{2xy} \]
\[ 38. \quad \frac{dy}{dx} = \frac{3y^2 - x^2}{2xy} \]

2.3 Modeling with First Order Equations

Differential equations are of interest to nonmathematicians primarily because of the possibility of using them to investigate a wide variety of problems in the physical, biological, and social sciences. One reason for this is that mathematical models and